A Proofs for Kaleidoscopic Orbifold Enumeration

In Table 1 we provide an enumeration of 2D kaleidoscopic orbifolds based on the cardinality of their underlying polygons and the type of their universal covers. To justify this enumeration, we organize the computation behind this enumeration into a number of theorems in this section.

Theorem 1. *Given a kaleidoscopic orbifold $O = *k_1k_2\ldots k_N$, O is a spherical orbifold if $N \le 2$ and a hyperbolic orbifold if $N > 4$. When $N = 4$, O is a hyperbolic orbifold with the only exception of $*2222$, which is a Euclidean orbifold.*

Proof. To show these statements, we only need to compute the Euler characteristics of these orbifolds using Equation 1 in the paper which states that Euler characteristic of a kaleidoscopic orbifold $O = *k_1k_2\ldots k_N$ is $\chi(O) = \sum_{i=1}^{N} \frac{1}{2k_i} - \frac{N}{2} + 1$.

When $N = 1$, the polygon is a monogon, with one mirror wall that self-intersects at an angle of $\frac{\pi}{k}$. The only good kaleidoscopic orbifold is when $k = 1$, i.e. the wall self-intersects at an angle of π. Note that the orbifold $O = *1$ is usually abbreviated as $*$. Thus, $\chi(O) = \frac{1}{2} - \frac{1}{2} + 1 = 1 > 0$. Consequently, this orbifold is spherical. In fact, this orbifold is a hemisphere with the boundary having the reflectional symmetry.

When $N = 2$, $O = *k_1k_2$ where $1 < k_1 \le k_2$. Consequently, $\chi(O) = \frac{1}{2k_1} + \frac{1}{2k_2} - \frac{2}{2} + 1 = \frac{1}{2k_1} + \frac{1}{2k_2} > 0$, which means that this type of orbifolds are also spherical.

On the other hand, when $N \ge 4$, we have $k_i \ge 2$ for any $1 \le i \le N$. Thus, $\chi(O) = \sum_{i=1}^{N} \frac{1}{2k_i} - \frac{N}{2} + 1 \le \frac{1}{4}N - \frac{N}{2} + 1 = 1 - \frac{N}{4} \le 0$. Notice that the equality holds in the above only when $N = 4$ and $k_1 = k_2 = k_3 = k_4 = 2$. Consequently, this type of orbifolds is hyperbolic with the only exception of $*2222$, which is a Euclidean orbifold.

□

Theorem 1 indicates that the more walls there are in the kaleidoscopic orbifold, the more negative its Euler characteristic and the more likely the orbifold being hyperbolic. In contrast, the fewer the walls the more positive its Euler characteristic and more likely the orbifold being spherical. The boundary between the set of spherical orbifolds and the set of hyperbolic orbifolds is drawn when $N = 3$, i.e. triangular orbifolds. The next several theorem inspect this scenario, which consists a number of cases.

Theorem 2. *Given a triangular kaleidoscopic orbifold $O = *k_1k_2k_3$ and without the loss of generality assuming that $3 \le k_1 \le k_2 \le k_3$, O is a hyperbolic orbifold with the only exception of $*333$, which is a Euclidean orbifold.*

Proof. First of all, the assumption that $3 \le k_1 \le k_2 \le k_3$ makes sense since any permutation of k_1, k_2, and k_3 gives rise the same triangular orbifold.

Again, we only need to compute the Euler characteristics of these orbifolds. Here, $\chi(O) = \sum_{i=1}^{3} \frac{1}{2k_i} - \frac{3}{2} + 1 \le \frac{1}{6}3 - \frac{3}{2} + 1 = 0$. Notice that the equality holds in the above only when $k_1 = k_2 = k_3 = 3$. Consequently, this type of orbifolds is hyperbolic with the only exception of $*333$, which is a Euclidean orbifold.

□

Theorem 2 states that for triangular orbifolds, the higher the minimal order of symmetry at the corners, namely k_1, the more likely the orbifold is hyperbolic. We now consider the case when $k_1 = 2$.

Theorem 3. *Given a triangular kaleidoscopic orbifold $O = *2k_2k_3$ where $2 \le k_2 \le k_3$, O is a spherical orbifold if $k_2 = 2$. In contrast, when $k_2 \ge 4$, O is a hyperbolic orbifold with the only exception of $*244$, which is a Euclidean orbifold.*

Proof. Since $N = 3$, when $k_1 = k_2 = 2$ we find the Euler characteristic $\chi(O) = \frac{1}{4} + \frac{1}{4} + \frac{1}{2k_3} - \frac{3}{2} + 1 = \frac{1}{2k_3} > 0$. Thus, in this case the orbifold is always spherical.

On the other hand, when $k_3 \ge k_2 \ge 4$, the Euler characteristics is $\chi(O) = \frac{1}{4} + \frac{1}{2k_2} + \frac{1}{2k_3} - \frac{3}{2} + 1 \le \frac{1}{4} + \frac{1}{8} + \frac{1}{8} - \frac{3}{2} + 1 = 0$. Notice that the equality holds in the above only when $k_2 = k_3 = 4$. Consequently, this type of orbifolds is hyperbolic with the only exception of $*244$, which is a Euclidean orbifold.

□

The last remaining case is when $O = *23k_3$, which is covered in the next theorem.

Theorem 4. *Given a triangular kaleidoscopic orbifold $O = *23k_3$ where $3 \le k_3$, O is a spherical orbifold if $k_3 < 6$, a Euclidean orbifold if $k_3 = 6$, and a hyperbolic orbifold if $k_3 > 6$.*

Proof. The Euler characteristic of this type of orbifolds is $\chi(O) = \frac{1}{4} + \frac{1}{6} + \frac{1}{2k_3} - \frac{3}{2} + 1 = \frac{1}{2k_3} - \frac{1}{12} = \frac{6-k_3}{12k_3}$. Thus, $\chi(O)$ is positive when $k_3 < 6$, zero when $k_3 = 6$, and negative when $k_3 > 6$. Consequently, O is a spherical orbifold if $k_3 < 6$, a Euclidean orbifold if $k_3 = 6$, and a hyperbolic orbifold if $k_3 > 6$.

□

Interestingly, each of the above theorems contains a Euclidean orbifold: $*2222$ for Theorem 1, $*333$ for Theorem 2, $*244$ for Theorem 3, and $*236$ for Theorem 4. Not only do these facts confirm that there are only four Euclidean kaleidoscopic orbifolds, but they also show the transition from spherical orbifolds to hyperbolic orbifolds with more walls and higher-order symmetries at the corners.

B Optics-Based Visualization for Orbifold Concept and Properties

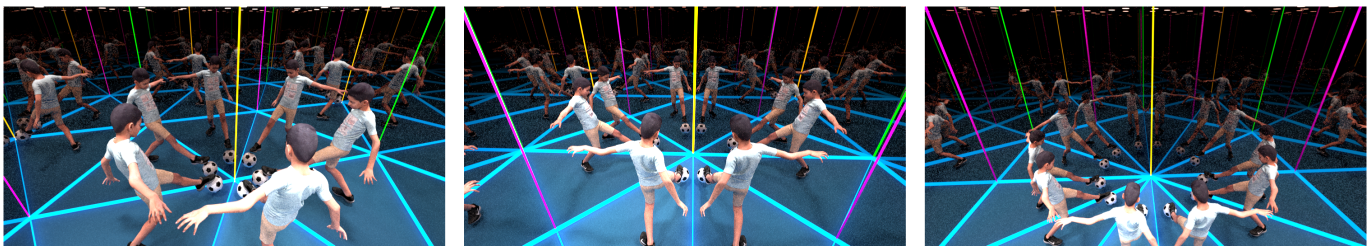
Our system can be used to generate example scenarios to illustrate important concepts and properties of orbifolds such as the following. Given a room with the statue Lucy, we first mount a mirror each on two adjacent walls (Figure 15 (a)). This leads to an illusion of a space that is four times as large as the room without a mirror. The virtual space is the *universal cover* of the orbifold (the original room).

In addition, the symmetry for the room can be understood by checking the orientations of the statues in the space. While the statue has her left hand up holding the torch in the original room, each mirror generates a virtual statue who raises the torch by her right hand (a reflection). Interestingly, reflecting the statue in the first virtual room with respect to the second mirror leads to the third virtual statue, who switches back to her left hand to raise the torch. However, this virtual statue faces the opposite direction of the statue in the original room, i.e. a rotation by π. One can consider the reflected and rotated virtual copies as the result of the *action* of the *symmetry group* of the underlying orbifold. This group consists of the identity action, two reflections (one per each mirror), and one rotation (the composite of the two mirrors).

By moving one of the mirrors to the wall opposite the other mirror, we obtain a different scene where there are infinitely many copies of

(a) **two adjacent mirrors** (b) **two parallel mirrors** (c) **four mirrors**

Fig. 15: A square room with two or four mirrors. The case in the four mirror room (c) corresponds to a Euclidean orbifold $*2222$.



(a) *333 (b) $*244$ (c) $*236$

Fig. 16: The three triangular Euclidean kaleidoscopic orbifolds.

the original room (Figure 15 (b)). In fact, the universal cover of this orbifold can be generated by first grouping the original room with one of the reflections and then translating infinitely many times the two rooms by a distance that is a multiple of twice the room depth. The union of the two rooms (the real room and the virtual room) is thus referred to as a *translational cover*.

When a mirror is mounted on each wall (Figure 15 (c)), we obtain the orbifold whose translational cover is the same as the universal cover of the room shown in Figure 15 (a). This translational cover is then translated in two mutually perpendicular directions. Note that this is the first orbifold (in this example) that we have encountered where all walls have a mirror. This room corresponds to the $*2222$ orbifold. Each of the corner has an angle of $\frac{\pi}{2}$, thus its notation. At such a corner, there are $2k$ copies of the original room forming k pairs. Inside each pair, one of the rooms is a rotational copy of the original room while the other is a reflectional copy. A *kaleidoscopic orbifold* has a *transformation group* that is generated by mirror reflections. The *subgroup* for each corner is thus $\mathbb{D}_k$, the *Dihedral group* of *order k*. In the $*2222$ case, the symmetry group at every corner is the same, i.e. $\mathbb{D}_2$.

$*2222$ is one of the four Euclidean kaleidoscopic orbifolds, i.e. whose universal cover is the Euclidean plane. Figure 16 shows the other three such orbifolds: (1) $*333$, (2) $*244$, and (3) $*236$. The $*333$ orbifold (Figure 16 (a)) is obtained by placing three mirror walls in a $\frac{\pi}{3}-\frac{\pi}{3}-\frac{\pi}{3}$ triangular room. Its translational cover consists of six copies of the original room ($\mathbb{D}_3$). Similarly, the $*244$ orbifold (Figure 16 (b)) is obtained by placing three mirror walls in a $\frac{\pi}{2}-\frac{\pi}{4}-\frac{\pi}{4}$ triangular room. Its translational cover consists of eight copies of the original room ($\mathbb{D}_4$). The $*236$ orbifold (Figure 16 (c)) is generated by placing three mirror walls in a $\frac{\pi}{2}-\frac{\pi}{3}-\frac{\pi}{6}$ triangular room. Its translational cover consists of 12 copies of the original room ($\mathbb{D}_6$). Notice that the symmetry group can vary from corner to corner. In addition, note that an orbifold does not depend on which corner is referred to as the first corner. Thus, $*236$ and $*362$ represent the same orbifold. Similarly, the orbifold does not change when the corners are numbered in the opposite order. Thus, $*236$ and $*632$ also represent the same orbifold.

The optics-based visual metaphor is also capable of showing non-Euclidean orbifolds, such as those shown in Figures 1, 2, and 14 in the main paper.