Tuning Continua and Keyboard Layouts

ANDREW MILNE*†, WILLIAM SETHARES‡, and JAMES PLAMONDON§

†Department of Music, P.O. Box 35, FIN-40014, University of Jyväskylä, Finland. Email: andymilne@tonalcentre.org
‡1415 Engineering Drive, Dept Electrical and Computer Engineering, University of Wisconsin, Madison, WI 53706 USA.
Phone: 608-262-5669. Email: sethares@ece.wisc.edu
§CEO, Thumtronics Inc., 6911 Thistle Hill Way, Austin, TX 78754 USA. Phone: 512-363-7094. Email:
jim@thumtronics.com

Previous work has demonstrated the existence of keyboard layouts capable of maintaining consistent fingerings across a parameterised family of tunings. This paper describes the general principles underlying layouts that are invariant in both transposition and tuning. Straightforward computational methods for determining appropriate bases for a regular temperament are given in terms of a row-reduced matrix for the temperament-mapping. A concrete description of the range over which consistent fingering can be maintained is described by the valid tuning range. Measures of the resulting keyboard layouts allow direct comparison of the ease with which various chordal and scalic patterns can be fingered as a function of the keyboard geometry. A number of concrete examples illustrate the generality of the methods and their applicability to a wide variety of commas and temperaments, tuning continua, and keyboard layouts.

Keywords: Temperament; Comma; Layout; Button-Lattice; Swathe; Isotone

1   Introduction

Some alternative keyboard designs have the property that any given interval is fingered the same in all keys. Recent work [1, 2] has shown by example that it is possible to have consistent fingering not only across all keys in a single tuning, but also across a range of tunings. For example, a 12-edo (equal divisions of the octave) major chord, a 19-edo major chord, and a Pythagorean major chord might all be fingered in the same way. This paper explores the scope of this idea by parameterising tunings using *regular temperaments* which simplify Just Intonations (JI) into systems that can be easily controlled and played.

As shown in Fig. 1, an underlying JI system is mapped to a set of generators of a reduced rank tuning system. Sect. 2 describes this mapping in terms of the null space of a collection of commas. The resulting generators of the reduced rank system are variable, and control the tuning of all notes at any given time. A privileged set of intervals (such as the primary consonances) is used in Sect. 3 to calculate the range over which the generator may vary and still retain consistent fingering. Sect. 4 describes a basis for the button-lattice which is chosen to spatially arrange the intervals of the temperament for easy playability. Issues contributing to the playability of an instrument include transpositional variance/invariance, the geometry of the swathe (discussed in Sect. 4), the presence of a monotonic pitch axis (detailed in Sect. 6), and readily fingerable harmonic and melodic intervals (Sect. 7). A series of examples in Sect. 5 shows how the ideas can be applied to a variety of tuning continua and a variety of keyboard geometries.

2   Tempering by commas

A p-dimensional tuning system can be tempered with n commas, providing a way to parameterise a family of tunings in terms of a smaller number (typically one or two) of basis elements. Suppose a system $\mathcal{S}$ is generated by

---

*Corresponding author. Email: andymilne@tonalcentre.org
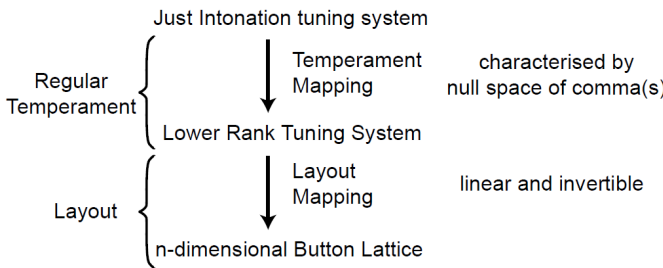
Milne, Sethares, Plamondon

Figure 1. From JI to regular temperament to layout.

p positive real generators $g_1, g_2, \ldots, g_p$, so that any element $s \in \mathcal{S}$ can be expressed uniquely as integer powers of the g_j, that is, $s = g_1^{i_1} g_2^{i_2} \cdots g_p^{i_p}$ for $i_j \in \mathbb{Z}$. The value of s can be interpreted as the frequency height of the p-dimensional lattice element $z = (i_1, i_2, \ldots i_p) \in \mathbb{Z}^p$. It is often easier to study such a system by taking the logarithm; this turns powers into products, products into sums, and frequency height into *pitch height* (or cents). In this notation, the *tuning vector* $g = (\log(g_1), \log(g_2), \ldots, \log(g_p)) \in \mathbb{R}^p$ maps each element $z \in \mathbb{Z}^p$ to

$$i_1 \log(g_1) + i_2 \log(g_2) + \ldots + i_p \log(g_p) = \langle z, g \rangle \in \mathbb{R}.$$

The value $\langle z, g \rangle$ is the pitch height of z corresponding to the tuning vector. *Tempering* means to vary the precise values of the tuning vector, replacing g with nearby values $G = (\log(G_1), \log(G_2), \ldots, \log(G_p)) \in \mathbb{R}^p$. The orthogonal complement

$$G^\perp = \{x \in \mathbb{R}^p | \langle G, x \rangle = 0\}$$

is the $p-1$ dimensional *isotone hyperplane* of pitch height zero. A *comma* is an element $c \in \mathbb{Z}^p$ with $\langle c, G \rangle = 0$ and so $G^\perp \bigcap \mathbb{Z}^p$ is called the *comma lattice*. A *tuning continuum* is a continuous family of tuning vectors $G_\gamma \in \mathbb{R}^p$ with the property that there is a common n-dimensional subspace $C \subset G_\gamma^\perp$ of the isotone hyperplanes for all members of the family. (Particular members of the continuum may have more than n commas.) A tuning continuum is called a *regular temperament* when its tuning gives intervals that can be correctly recognised as approximations of small integer (Just Intonation) ratios. A simple way to concretely characterise a tuning continuum is in terms of a basis for C, that is, in terms of n linearly independent commas. The commas can be viewed as a set of constraints that reduce the system from p to $p - n$ dimensions.

Let $c_1, c_2, \ldots, c_n \in \mathbb{Z}^p$ be n linearly independent commas gathered into a matrix $\mathbf{C} \in \mathbb{R}^{n \times p}$ whose jth row is c_j. Since $\langle c_j, G \rangle = 0$ for all j, this represents a linear system of equations

$$\mathbf{C} (\log(G_1), \log(G_2), \ldots, \log(G_p))^{\mathrm{T}} = 0$$

(where x^{T} is the transpose of x) and the mapping can be characterised by the null space $\mathcal{N}(\mathbf{C})$. The corresponding range space mapping $\mathbf{R} : \mathbb{Z}^p \to \mathbb{Z}^n$ (defined by the transpose of some basis of $\mathcal{N}(\mathbf{C})$) maps any element $g_1^{i_1} g_2^{i_2} \cdots g_p^{i_p} \in \mathcal{S}$ to

$$\mathbf{R} (i_1, i_2, \ldots, i_p)^{\mathrm{T}}.$$

Such characterisations of temperaments in terms of the null space of commas were first discussed on the Alternate Tunings Mailing List [3] and a variety of special cases are considered in [4] and [5]. A basis for the temperament mapping can often be found by reducing $\mathbf{R}$ to row echelon form $\hat{\mathbf{R}}$ which displays the same range space with leading '1's. Any such basis can be used to specify the tuning continuum.

EXAMPLE 2.1 (5-limit with the Syntonic Comma and Major Diesis) *Consider the 5-limit primes defined by the generators* 2, 3, 5, *which are tempered to* $2 \mapsto G_1$, $3 \mapsto G_2$, *and* $5 \mapsto G_3$ *by the* syntonic comma $G_1^{-4} G_2^4 G_3^{-1} = 1$ *and the* major diesis $G_1^3 G_2^4 G_3^{-4} = 1$. *Then* $\mathbf{C} = \left(\begin{smallmatrix} -4 & 4 & -1 \\ 3 & 4 & -4 \end{smallmatrix} \right)$ *has null space* $\mathcal{N}(\mathbf{C}) = (12, 19, 28)^{\mathrm{T}}$. *Thus* $\mathbf{R} = (12, 19, 28)$, *and a typical element* $2^{i_1} 3^{i_2} 5^{i_3}$ *is tempered to* $G_1^{i_1} G_2^{i_2} G_3^{i_3}$ *and then mapped through the comma to*

$12i_1 + 19i_2 + 28i_3$. *All three temperings can be written in terms of a single variable α as $G_1 = \alpha^{12}$, $G_2 = \alpha^{19}$, and $G_3 = \alpha^{28}$. If the choice is made to temper G_1 to 2 (to leave the octave unchanged) $\alpha = \sqrt[12]{2}$ and the result is 12-edo. If G_1 is tempered to a* pseudo-octave *p_e, $\alpha = \sqrt[12]{p_e}$ defines the tuning with 12 equal divisions of the pseudo-octave [6, 7]. Any given α defines a member (i.e. a particular tuning) within this 12-equal divisions of the pseudo-octave tuning continuum.*

EXAMPLE 2.2 (5-limit with the Syntonic Comma) *Consider the 5-limit primes defined by the generators 2, 3, 5, which are tempered to $2 \mapsto G_1$, $3 \mapsto G_2$, and $5 \mapsto G_3$ by the syntonic comma $G_1^{-4} G_2^4 G_3^{-1} = 1$. Then $\mathbf{C} = (-4, 4, -1)$ has null space spanned by the rows of $\mathbf{R} = \left(\begin{smallmatrix} 1 & 1 & 0 \\ -1 & 0 & 4 \end{smallmatrix} \right)$, and this may be row reduced to*

$$\hat{\mathbf{R}} = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 4 \end{pmatrix}. \tag{1}$$

A typical element $(i_1, i_2, i_3)^{\mathrm{T}}$ is mapped to $(i_1 - 4i_3, i_2 + 4i_3)^{\mathrm{T}}$. In this basis, the tempered generators can be written in terms of two basis elements $\hat{\alpha}$ and $\hat{\beta}$ by inspection of the columns of $\hat{\mathbf{R}}$ as $G_1 = \hat{\alpha}$, $G_2 = \hat{\beta}$, and $G_3 = \hat{\alpha}^{-4} \hat{\beta}^4$. Tempering G_1 to 2 (= $\hat{\alpha}$, leaving the octave unchanged) and letting $\hat{\beta}$ vary gives a tuning continuum that contains a variety of well known tunings: $\hat{\beta} = 2^{\frac{11}{7}} \approx 2.9719$ gives 7-edo, $\hat{\beta} = 2^{\frac{19}{12}} \approx 2.9966$ gives 12-edo, $\hat{\beta} = 2^{\frac{27}{17}} \approx 3.0062$ gives 17-edo, $\hat{\beta} = 2^{\frac{8}{5}} \approx 3.0314$ gives 5-edo, and other $\hat{\beta}$ give a variety of (nonequal) tunings such as $\frac{1}{4}$-comma and $\frac{2}{7}$-comma meantone. If G_1 is tempered to a pseudo-octave $p_e = \hat{\alpha}$, the various values of $\hat{\beta}$ correspond to 7-equal division of the pseudo-octave, 12-equal divisions of the pseudo-octave, etc.

EXAMPLE 2.3 (5-limit with the Syntonic Comma) *Consider the 5-limit primes defined by the generators 2, $\frac{3}{2}$, $\frac{5}{4}$, which are tempered to $2 \mapsto H_1$, $\frac{3}{2} \mapsto H_2$, and $\frac{5}{4} \mapsto H_3$ by the syntonic comma $H_1^{-2} H_2^4 H_3^{-1} = 1$. Then $\mathbf{C} = (-2, 4, 1)$ has null space spanned by the rows of*

$$\hat{\mathbf{R}} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \end{pmatrix} \tag{2}$$

which is given in row reduced form. A typical element $(i_1, i_2, i_3)^{\mathrm{T}}$ is mapped to $(i_1 - 2i_3, i_2 + 4i_3)^{\mathrm{T}}$. In this basis, the tempered generators can be written in terms of two basis elements $\hat{\alpha}$ and $\hat{\beta}$ by inspection of the columns of $\hat{\mathbf{R}}$ as $H_1 = \hat{\alpha}$, $H_2 = \hat{\beta}$, and $H_3 = \hat{\alpha}^{-2} \hat{\beta}^4$. Tempering G_1 to 2, $\hat{\beta}$ covers the same gamut of tunings as in Example 2.2. For instance, as $\hat{\beta}$ increases, the continuum in Example 2.2 moves from 7, to 12, to 17, to 5-edo.

EXAMPLE 2.4 (5-limit with Two Commas) *Consider the 5-limit primes defined by the generators 2, 3, 5, tempered to $2 \mapsto G_1$, $3 \mapsto G_2$, and $5 \mapsto G_3$ by any two of the following commas:*

Name	Comma	Vector Representation
the syntonic comma	$G_1^{-4} G_2^4 G_3^{-1} = 1$	$(-4, 4, -1)$
the parakleisma	$G_1^8 G_2^{14} G_3^{-13} = 1$	$(8, 14, -13)$
the kleisma	$G_1^{-6} G_2^{-5} G_3^6 = 1$	$(-6, -5, 6)$
the small diesis	$G_1^{-10} G_2^{-1} G_3^5 = 1$	$(-10, -1, 5)$

Then $\mathbf{C}$ is a 2x3 matrix composed of any two of the vectors above. All pairs have the same null space $\mathcal{N}(\mathbf{C}) = (19, 30, 44)^{\mathrm{T}}$ which is mapped through the comma to $19i_1 + 30i_2 + 44i_3$. The temperings can be written in terms of a single variable α as $G_1 = \alpha^{19}$, $G_2 = \alpha^{30}$, and $G_3 = \alpha^{44}$. If the choice is made to temper G_1 to 2 (to leave the octave unchanged) then $\alpha = \sqrt[19]{2}$ and the result is 19-edo. Similarly, tempering G_1 to a pseudo-octave p_e defines the tuning continuum with 19-equal divisions of the pseudo-octave.

EXAMPLE 2.5 (11-limit) *Consider the 11-limit primes defined by the generators 2, 3, 5, 7, 11, which are tempered*

to $2 \mapsto G_1$, $3 \mapsto G_2$, $5 \mapsto G_3$, $7 \mapsto G_4$, *and* $11 \mapsto G_5$. *Using the three commas*

$$G_1^{-7} G_2^{-1} G_3^1 G_4^1 G_5^1 = 1$$

$$G_1^1 G_2^2 G_3^{-3} G_4^1 = 1$$

$$G_1^{-4} G_2^4 G_3^1 = 1$$

reduces this to a rank 2 regular tuning system. The matrix $\mathbf{C} = \begin{pmatrix} -7 & -1 & 1 & 1 & 0 \\ 1 & 2 & -3 & 1 & 0 \\ -4 & 4 & 1 & 0 & 0 \end{pmatrix}$ *has null space* $\mathcal{N}(\mathbf{C}) = \begin{pmatrix} 13 & 24 & 44 & 71 & 0 \\ 10 & 13 & 12 & 0 & 71 \end{pmatrix}^{\mathrm{T}}$. *The row reduced form of* $\mathbf{R}$ *is* $\hat{\mathbf{R}} = \begin{pmatrix} 1 & 0 & -4 & -13 & 24 \\ 0 & 1 & 4 & 10 & -13 \end{pmatrix}$, *which implies the basis* $G_1 = \hat{\alpha}$, $G_2 = \hat{\beta}$, $G_3 = \hat{\alpha}^{-4} \hat{\beta}^4$, $G_4 = \hat{\alpha}^{-13} \hat{\beta}^{10}$, *and* $G_5 = \hat{\alpha}^{24} \hat{\beta}^{-13}$.

Examples 2.2, 2.3, and 2.5 define two-parameter mappings called α-*reduced* β-*chains*, which are generated by stacking integer powers of β and then *reducing* (dividing or multiplying by α) so that every term lies between 1 and α. For any $i \in \mathbb{Z}$, this is $\beta^i \alpha^{-\lfloor i \log_\alpha(\beta) \rfloor}$ where $\lfloor x \rfloor$ represents the largest integer less than or equal to x. α-reduced β-chains define scales that repeat at intervals of α; $\alpha = 2$, representing repetition at the octave, is the most common value. For a chain to repeat at the octave, $2^{\frac{1}{r}} \mapsto \alpha$ for some $r \in \mathbb{Z}$. Similarly, for a chain to repeat at another interval of equivalence p_e (such as a 'stretched octave' $p_e = 2.01$, the 'tritave' $p_e = 3$, or the 'pentave' $p_e = 5$), $p_e^{\frac{1}{r}} \mapsto \alpha$. If $\alpha^m = \beta^n$ for some coprime integers m and n, the chain has n distinct notes in $[1, \alpha)$. If α and β are not rationally related, there are an infinite number of notes dense in $[1, \alpha)$. Any arbitrary segment of an α-reduced β-chain can be used to form a scale, and the number of notes it contains is called its *cardinality*. For a given tuning of α and β, certain cardinalities are *MOS scales*, which have a number of musically advantageous properties (when $\alpha = 2$, MOS scales are conceptually equivalent to *well-formed* scales) [8–10].

3   Valid tuning range for consistent fingering

A p-dimensional tuning system tempered by n commas parameterises a $p - n$ dimensional family of tunings. When such a tuning system is laid out onto a keyboard, two desirable properties are *transpositional invariance* (where intervals are fingered the same in all keys) and *tuning invariance* (where intervals are fingered the same throughout all members of a tuning continuum). As shown in [1], these together require the keyboard to have dimension $p - n$. This section generalises these ideas by formally defining *consistent fingering of a set of intervals* over the *valid tuning range* where the fingering of intervals on the keyboard remains the same.

Any interval $s \in \mathcal{S}$ can be written in terms of the p generators as $s = g_1^{i_1} g_2^{i_2} \cdots g_p^{i_p}$ where $i_j \in \mathbb{Z}$, or more concisely as the vector $\boldsymbol{s} = (i_1, i_2, \cdots i_p)$. As in Sect. 2, a set of n commas defines the temperament mapping R. Consider a *privileged* set of intervals

$$1 = s_0 < s_1 < s_2 < \ldots < s_Q, \tag{3}$$

which can also be represented as the vectors $\boldsymbol{s}_0, \boldsymbol{s}_1, \ldots, \boldsymbol{s}_Q$. Given any set of generators $\alpha_1, \alpha_2, \ldots, \alpha_{p-n}$ for the reduced rank tuning system, each interval $\boldsymbol{s}_i$ is tempered to $\mathbf{R}\boldsymbol{s}_i = \alpha_1^{j_1} \alpha_2^{j_2} \ldots \alpha_{p-n}^{j_{p-n}}$ where $j_k \in \mathbb{Z}$. Without loss of generality, the α_k may be assumed greater than or equal to unity (since a candidate generator $\alpha_k < 1$ with exponent j_k is equivalent to the generator $\alpha_k^{-1} > 1$ with exponent $-j_k$). The α-generators define the specific tempered tuning and the coefficients j_k specify the exact ratios of the privileged intervals within the temperament. The set of all generators α_i for which

$$1 = \mathbf{R}s_0 < \mathbf{R}s_1 < \mathbf{R}s_2 < \ldots < \mathbf{R}s_Q \tag{4}$$

holds is called the *valid tuning range* (VTR). If the musical system contains an interval of equivalence p_e, the privileged intervals need only be considered in p_e-*reduced* form.

Given any layout mapping $\mathbf{L}$ (as discussed in Sect. 4), the j_k correspond directly to the buttons that must be pressed in order to sound those intervals. Accordingly, (4) guarantees that the fingering of privileged intervals

remains fixed for all possible perturbations of the basis within the VTR and the privileged intervals are said to be *consistently fingered* over the VTR. Note that the VTR does not guarantee that the privileged intervals sound 'in tune' or remain recognisable in a harmonic sense (i.e. as equivalent to specific small-integer ratios), but it does guarantee that the scalic order of all privileged intervals is invariant. For example, a scale (e.g. the chromatic) might consist of privileged intervals (e.g. the harmonic consonances of common practice) and the intervals that can connect these privileged intervals by all possible types of contrapuntal motion (i.e. similar and contrary and oblique) (e.g. the tones and semitones of common practice). These connective intervals may serve as the raw material for melodies and for voice-leading between harmonies, and so support a system of counterpoint. In such a scale, the melody will have the same contour (i.e., the same pattern of up and down notes) for all tunings in the continuum within the VTR. Outside the VTR, the contour, and hence the melodies, will change. Thus the VTR defines the limit over which the system of counterpoint remains invariant (of course, the adventurous composer or performer may choose to explore the exotic melodic terrain beyond the VTR).

For non-contrapuntal systems it may be more useful to choose privileged intervals that are not harmonic consonances but are instead important melodic entities, such as the two differently sized steps of an MOS scale (see Example 3.5, for a concrete example). In this way, the VTR of any given MOS scale (characterised according to its cardinality) is equivalent to the tuning range over which that scale can actually exist. In such a system, the melodic contour (and, therefore fingering) of all within-scale playing is invariant throughout the VTR.

There is an interesting relationship between the VTRs of some regular temperaments (using harmonic consonances as the privileged intervals) and the VTR of an associated MOS scale (using that scale's steps as privileged intervals). When the VTR of the MOS scale includes the VTR of the regular temperament, it implies that all of that MOS's melodic intervals can also voice-lead between all of that temperament's consonant harmonies. Such systems might be considered to have desirable musical properties because their melodic and harmonic structures are so tightly bound together. Example 3.5 gives a concrete example: the seven note diatonic MOS scale used to embody the syntonic temperament (of which meantone tunings are the most familiar) is a system in which every melodic interval is also a voice-leading interval. An exploration of other such harmonic/melodic synergies, and the relationships between VTRs and coherence [11], may provide interesting topics for future investigations.

EXAMPLE 3.1 (The Primary Consonances) *Perhaps the most common example of a privileged set of intervals in 5-limit JI (recall Examples 2.1-2.3) is the set of eight common practice consonant intervals* $1, \frac{6}{5}, \frac{5}{4}, \frac{4}{3}, \frac{3}{2}, \frac{8}{5}, \frac{5}{3}, 2$, *which are familiar to musicians as the unison, just major and minor thirds, just perfect fifth, octave, and their octave inversions.*

EXAMPLE 3.2 (Higher-Limit Primary Consonances) *The* major prime chord *of a p-limit just intonation is built from an octave-reduced version of the primes* $2 : 3 : 5 : \ldots : p$. *The* minor prime chord *is analogously built from octave-reduced versions of the inverted primes* $\frac{1}{2} : \frac{1}{3} : \frac{1}{5} : \ldots : \frac{1}{p}$. *The intervals that make up these chords form natural candidates for the privileged set when working with higher order just intonations.*

Given a tuning system $\mathcal{S}$, a set of commas and a set of privileged intervals, it is important to be able to find the VTR. After tempering, each of the privileged intervals $\mathbf{R}s_i$ may be represented as an element $s_i \in \mathbb{Z}^{p-n}$. The requirement that $\mathbf{R}s_{i+1} > \mathbf{R}s_i$ in (4) is identical to the requirement that

$$\begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_{p-n} \end{pmatrix} \begin{pmatrix} \log(\alpha_1) \\ \log(\alpha_2) \\ \vdots \\ \log(\alpha_{p-n}) \end{pmatrix} > \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_{p-n-1} \end{pmatrix} \begin{pmatrix} \log(\alpha_1) \\ \log(\alpha_2) \\ \vdots \\ \log(\alpha_{p-n}) \end{pmatrix} \tag{5}$$

where the inequality signifies an element-by-element operation. Let

$$\mathbf{S} = \begin{pmatrix} s_1 - s_0 \\ s_2 - s_1 \\ \vdots \\ s_{p-n} - s_{p-n-1} \end{pmatrix} \text{ and } x = \begin{pmatrix} \log(\alpha_1) \\ \log(\alpha_2) \\ \vdots \\ \log(\alpha_{p-n}) \end{pmatrix}. \tag{6}$$

Then (5) becomes $\mathbf{S}x > 0$, which is the intersection of $p - n$ half-planes with boundaries that pass through the origin. Since the elements of x are positive (which follows because $\alpha_i \geq 1$), only the first quadrant need be considered. The monotonicity assumption (4) guarantees that the intersection is a nonempty cone radiating from the origin; this cone defines the VTR.

EXAMPLE 3.3 (5-Limit Syntonic Continuum) *Consider the* 5*-limit JI system in Example 2.2 in which the syntonic comma is used to give the temperament mapping* $\hat{\mathbf{R}}$ *in (1). The primary consonances of Example 3.1 are mapped through* $\hat{\mathbf{R}}$ *to* $\left(\begin{smallmatrix} 0 & 5 & -6 & 2 & -1 & 7 & -4 & 1 \\ 0 & -3 & 4 & -1 & 1 & -4 & 3 & 0 \end{smallmatrix}\right)^{\mathrm{T}}$ *which can be read in terms of two generators* α *and* β *(renamed from* α_1 *and* α_2 *above) as a requirement that the inequalities*

$$1 = \alpha^0\beta^0 < \alpha^5\beta^{-3} < \alpha^{-6}\beta^4 < \alpha^2\beta^{-1} < \alpha^{-1}\beta^1 < \alpha^7\beta^{-4} < \alpha^{-4}\beta^3 < \alpha^1\beta^0$$

hold for all α *and* β *in the VTR. Rewriting this as in (5)-(6) gives*

$$\begin{pmatrix} 5 & -11 & 8 & -3 & 8 & -11 & 5 \\ -3 & 7 & -5 & 2 & -5 & 7 & -3 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} > \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This region is the cone bounded below by $x_2 = \frac{11}{7}x_1$ *and bounded above by* $x_2 = \frac{8}{5}x_1$*. Since* $x_1 = \log(\alpha)$ *and* $x_2 = \log(\beta)$*, these can be solved by exponentiation to show that* $\alpha^{\frac{11}{7}} < \beta < \alpha^{\frac{8}{5}}$*. For* $\alpha = 2$*, this covers the range between 7-edo and 5-edo. Outside this range, one or more of the privileged intervals changes fingering. At the boundaries, the difference between two of the privileged intervals collapses to a unison. This VTR range is identical to Blackwood's range of recognisable diatonic tunings [12], to the 12-note MOS scale generated by fifths, and to the range of the syntonic tuning continuum in [1, 2]. The syntonic tuning continuum contains the familiar meantone tunings used in common practice, though the term meantone typically refers to the narrower range of syntonic tunings that have an established historical usage (a range of approximately 19-edo to 12-edo). An MOS scale containing only the primary consonances and the voice-leading intervals that connect them is the 12-note chromatic, and hence the contour of any chromatic melody (or diatonic, because it is a subset of the chromatic) is preserved over all members of the syntonic continuum.*

EXAMPLE 3.4 (5-Limit Syntonic Continuum II) *The* 5*-limit JI system in Example 2.3 is defined by the generators* 2*,* $\frac{3}{2}$*,* $\frac{5}{4}$*, uses the syntonic comma, and has temperament map* $\hat{\mathbf{R}}$ *in (2). The primary consonances of Example 3.1 are mapped through* $\hat{\mathbf{R}}$ *to* $\left(\begin{smallmatrix} 0 & 2 & -2 & 1 & 0 & 3 & -1 & 1 \\ 0 & -3 & 4 & -1 & 1 & -4 & 3 & 0 \end{smallmatrix}\right)^{\mathrm{T}}$*, which can be rewritten using (5)-(6) as*

$$\begin{pmatrix} 2 & -4 & 3 & -1 & 3 & -4 & 2 \\ -3 & 7 & -5 & 2 & -5 & 7 & -3 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} > \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This region is the cone bounded below by $x_2 = \frac{4}{7}x_1$ *and bounded above by* $x_2 = \frac{3}{5}x_1$*. This corresponds to the region* $\alpha^{\frac{4}{7}} < \beta < \alpha^{\frac{3}{5}}$ *which is again identical to Blackwood's range of recognisable diatonic tunings and to the 12-note MOS scale generated by fifths.*

The above examples show how the VTR changes when the temperament is expressed in different bases. Different privileged intervals imply different VTRs. For example, if some of the intervals are removed from consideration in Examples 3.3 and 3.4, the VTR may expand. If more intervals are included in the privileged set, the VTR may contract.

EXAMPLE 3.5 (The Seven-Note MOS Scale Generated by "Fifths") *Perhaps the most familiar example of a privileged set of melodic intervals are the tones and semitones that make up the diatonic scale. When* $2 \mapsto G_1$*,* $3/2 \mapsto G_2$*, the diatonic steps are* $\left(\begin{smallmatrix} 0 & -1 & 3 & 1 \\ 0 & 2 & -5 & 0 \end{smallmatrix}\right)^{\mathrm{T}}$*. Rewriting this as in (5)-(6) gives* $\left(\begin{smallmatrix} -1 & 4 & -2 \\ 2 & -7 & 5 \end{smallmatrix}\right)^{\mathrm{T}} \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) > \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$*. This region is the cone bounded below by* $x_2 = \frac{1}{2}x_1$ *and bounded above by* $x_2 = \frac{4}{7}x_1$*, corresponding to the region* $\alpha^{\frac{1}{2}} < \beta < \alpha^{\frac{4}{7}}$*, which includes the above syntonic VTR.*

EXAMPLE 3.6 (7-Limit Bohlen-Pierce) *A* 7*-limit Bohlen-Pierce system has odd generators* 3*,* 5*, and* 7 *which are tempered to* $3 \mapsto G_1$*,* $5 \mapsto G_2$*, and* $7 \mapsto G_3$ *by the comma* $G_1^{-7}G_2^6G_3^{-1} = 1$*.* $\mathbf{C} = (-7, 6, -1)$ *has null space*

Table 1.   A selection of temperaments with $2 \mapsto \alpha$ and $1 < \beta < 2^{\frac{1}{2}}$. With the exception of 'syntonic', the common names are taken from the Tonalsoft Encyclopedia of Microtonal Music Theory [13]. VTR values are rounded to the nearest cent and the comma vectors presume that $2 \mapsto G_1, 3 \mapsto G_2, 5 \mapsto G_3$.

Common name	Negri	Porcupine	Tetracot	Hanson	Magic	Würschmidt	Semisixths	Schismatic	Syntonic
Comma	$(-14,3,4)$	$(1,-5,3)$	$(5,-9,4)$	$(-6,-5,6)$	$(-10,-1,5)$	$(17,1,-8)$	$(2,9,-7)$	$(-15,8,1)$	$(-4,4,-1)$
VTR (cents)	120–150	150–171	171–185	300–327	360–400	375–400	436–450	494–514	480–514

spanned by the rows of $\mathbf{R} = \begin{pmatrix} 1 & 0 & -7 \\ 0 & 1 & 6 \end{pmatrix}$*, which is presented in row-reduced form. The privileged intervals (the analog of the primary consonances in the BP system) are* $1, \frac{9}{7}, \frac{7}{5}, \frac{5}{3}, \frac{9}{5}, \frac{15}{7}, \frac{7}{3}, 3$*, which are mapped through* $\mathbf{R}$ *to* $\begin{pmatrix} 0 & 9 & -7 & -1 & 2 & 8 & -8 & 1 \\ 0 & -6 & 5 & 1 & -1 & -5 & 6 & 0 \end{pmatrix}^{\mathrm{T}}$*. Rewriting this as in (5)-(6) gives* $\begin{pmatrix} 9 & -16 & 6 & 3 & 6 & -16 & 9 \\ -6 & 11 & -4 & -2 & -4 & 11 & -6 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} > \begin{pmatrix} 0 \\ 0 \end{pmatrix}$*. This region is the cone bounded below by* $x_2 = \frac{16}{11}x_1$ *and bounded above by* $x_2 = \frac{3}{2}x_1$*, which corresponds to the region* $\alpha^{\frac{16}{11}} < \beta < \alpha^{\frac{3}{2}}$ *and the tuning range of the 13-note MOS scale generated by* $\alpha \approx 3$ *and* $\beta \approx 5$*.*

EXAMPLE 3.7 (Reflected Scales) *Consider a basis expressed in the form* $\alpha = A$, $\beta = A^k B$ *where* $\alpha, \beta \geqslant 1$*. For any* $k \in \mathbb{Z}$*, all such bases produce identical* α*-reduced* β*-chains and identical VTRs (after* α*-reduction). The reflected basis* $\alpha = A$, $\beta = A^k B^{-1}$ *produces an* α*-reduced* β*-chain that is reflected about the zeroth note and a VTR that is reflected about* $\alpha^{\frac{1}{2}}$*. For instance a chain of six perfect fifth meantone generators produces the Lydian mode, which has step intervals of M2, M2, M2, m2, M2, M2, m2, and a VTR of* 686–720 *cents. A chain of six perfect fourth meantone generators produces the Phrygian mode, which has the same step intervals as Lydian but in reverse order (m2, M2, M2, m2, M2, M2, M2), and a VTR of* 480–514 *cents. Using the 5-limit primary consonances as the privileged intervals and fixing* α *at 2, Table 1 shows the* β*-tunings that are within the VTRs of a selection of temperaments (the tuning range of* β *is limited to* 1–$2^{\frac{1}{2}}$*, i.e.* 0–600 *cents).*

When assigning the values of α and β to controllable parameters, it is necessary to choose sensible ranges over which the parameters may vary: one strategy is to fix one component (say $\alpha = p_e^{1/r}$, thereby ensuring that the interval of equivalence is always purely tuned) and assign the other to a one-axis controller such as a slider. The performer moves the controller to change the tuning of the instrument, allowing exploration of the tunings within a given VTR and enabling easy transitions between different VTRs. Because all bases of the form $\alpha = A$, $\beta = A^k B$ are interchangeable (see Example 3.7), the tuning range of β need be no wider than α.

4   Layout mappings and the geometry of the swathe

A *button* is any device capable of triggering a specific pitch; it could be a physical object such as a key or lever, or it might be a 'virtual' object such as a position on a touch-sensitive display screen or in a holographic projection. A *layout* is the embodiment of a temperament in the button-lattice of a musical instrument. In the same way that a regular temperament has a finite number of generators (e.g. α and β) that generate its intervals, a lattice can be represented with a finite number of basis vectors that span its surface. The logical means to map from a temperament to a button-lattice is to map the generating intervals of the temperament to a basis of the lattice. The lattice generated by a full-rank matrix $\mathbf{L}$ is $\Lambda(\mathbf{L}) = \{\mathbf{L}k | k \in \mathbb{Z}^n\}$. If a copy of the fundamental parallelepiped $P(\mathbf{L}) = \{\mathbf{L}x | x \in \mathbb{R}^n, 0 \leq x_i < 1\}$ is placed at every point of the lattice, $\mathbb{R}^n$ is tiled by the parallelepipeds, each with volume $|\det(\mathbf{L})|$. The determinant of $\mathbf{L}$ is inversely proportional to the density of the lattice, that is, the number of lattice points inside a sphere O approaches $\frac{\mathrm{vol}(O)}{|\det(\mathbf{L})|}$ as O grows [14]. All layout mappings in this paper are assumed to have a determinant ± 1, which ensures that their button densities are equivalent and so comparisons between them are fair. Since transpositional invariance requires a linear and invertible layout mapping [1], a rank-2 temperament must be embodied on a 2-dimensional lattice. Thus most examples are drawn in two dimensions, where $\mathbf{L} : \mathbb{Z}^2 \to \mathbb{R}^2$ maps from the exponents i and j of the generators α^i and β^j through the layout mapping

$$\mathbf{L} = (\boldsymbol{\psi} \; \boldsymbol{\omega}) = \begin{pmatrix} \psi_x & \omega_x \\ \psi_y & \omega_y \end{pmatrix} \tag{7}$$

to the button-lattice. Physically, $\boldsymbol{\psi}$ and $\boldsymbol{\omega}$ form a basis of the button-lattice, which ensures that all notes $\alpha^i \beta^j$ are mapped to some button and that each button location corresponds to some note.

Table 2.   Matrix representations of a selection of button-lattices **B**, layouts **L**, and transformations **T** (rotation, reflection, scale, and shear). The values are given to two decimal places; the actual values for the layout matrices are multiples of the $\mathbf{B}_{\text{Hex}}$ vectors.

$$\text{Hexagonal: } \mathbf{B}_{\text{Hex}} = \begin{pmatrix} 2^{\frac{1}{2}}3^{-\frac{1}{4}} \approx 1.07 & 2^{-\frac{1}{2}}3^{-\frac{1}{4}} \approx 0.54 \\ 0 & 2^{-\frac{1}{2}}3^{\frac{1}{4}} \approx 0.93 \end{pmatrix} \quad \text{Square: } \mathbf{B}_{\text{Squ}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{Thummer: } \mathbf{B}_{\text{Thu}} = \begin{pmatrix} 1.25 & 0.62 \\ 0 & 0.80 \end{pmatrix} \quad \text{Wilson: } \mathbf{B}_{\text{Wil}} = \begin{pmatrix} 0.94 & 0.47 \\ 0.35 & 1.23 \end{pmatrix}$$

$$\mathbf{L}_1 = \begin{pmatrix} 0.54 & 1.07 \\ 0.93 & 0 \end{pmatrix} \quad \text{Wicki: } \mathbf{L}_{\text{Wic}} = \begin{pmatrix} 0 & 0.54 \\ 1.86 & 0.93 \end{pmatrix} \quad \mathbf{L}_2 = \begin{pmatrix} 2.69 & 1.61 \\ 0.93 & 0.93 \end{pmatrix} \quad \text{CBA-B: } \mathbf{L}_{\text{CBA-B}} = \begin{pmatrix} 3.76 & 2.15 \\ 2.79 & 1.86 \end{pmatrix}$$

$$\text{CBA-C: } \mathbf{L}_{\text{CBA-C}} = \begin{pmatrix} 3.76 & 2.15 \\ -2.79 & -1.86 \end{pmatrix} \quad \text{Fokker: } \mathbf{L}_{\text{Fok}} = \begin{pmatrix} 6.45 & 3.76 \\ 1.86 & 0.93 \end{pmatrix} \quad \text{Bosanquet: } \mathbf{L}_{\text{Bos}} = \begin{pmatrix} 4.90 & 2.86 \\ 0 & 0.20 \end{pmatrix} \quad \text{Wilson: } \mathbf{L}_{\text{Wil}} = \begin{pmatrix} 5.66 & 3.30 \\ 0 & 0.18 \end{pmatrix}$$

$$\mathbf{T}_{\text{Rot}}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \mathbf{T}_{\text{Ref}}(\theta) = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \quad \mathbf{T}_{\text{Sca}}(k) = \begin{pmatrix} k^{\frac{1}{2}} & 0 \\ 0 & k^{-\frac{1}{2}} \end{pmatrix} \quad \mathbf{T}_{\text{She}}(k_x,k_y) = \begin{pmatrix} 1+k_x k_y & k_x \\ k_y & 1 \end{pmatrix}$$

Table 2 shows the matrix representations of the reduced bases of a selection of unimodular lattices. The hexagonal button-lattice $\mathbf{B}_{\text{Hex}}$ allows for the densest possible packing of buttons [15]. For this reason many existing button-lattice instruments are approximately hexagonal in form: Thumtronics' Thummer [16], Starr Labs' Vath 648 [17], C-Thru Music's Axis [18], H-Pi's Tonal Plexus [19], the Fokker organ [20], and the generic chromatic button accordion. Table 2 also shows typical matrix representations for several common layouts [20–23]. As generic names, these may also refer to any reasonably similar layout. All values are given with respect to a hexagonal lattice, except for $\mathbf{L}_{\text{Bos}}$ and $\mathbf{L}_{\text{Wil}}$, which use the lattice forms in [23]. The syntonic tuning continuum of Example 2.3 is shown graphically in [1] as laid out with a Wicki mapping.

Left-multiplying a layout **L** by a unimodular real matrix may be interpreted as a transformation of the underlying lattice geometry and a selection of easy to visualise transformations is shown in Table 2. For example, $\mathbf{T}_{\text{Sca}}(1.346)*\mathbf{L}_{\text{Wic}}$ gives the Wicki layout mapping as applied to a Thummer lattice. Right-multiplying a layout **L** by a unimodular integer matrix may be interpreted as a new layout mapping to a different basis of the same underlying lattice. For example, $\mathbf{L}_{\text{Wic}} * \begin{pmatrix} -2 & -1 \\ 7 & 4 \end{pmatrix} = \mathbf{L}_{\text{CBA-B}}$.

As successive notes in an α-reduced β-chain are laid onto a button-lattice they cut a *swathe* across it. When α is the interval of equivalence p_e, it is musically meaningful to divide the intervals of a regular temperament into those that lie within a swathe (i.e. all α-reduced intervals larger than unison but smaller than α) and the repetitions of those intervals that lie outside that swathe (i.e. all intervals larger than α). The size of the swathe determines the microtonal and modulatory capabilities of the instrument; the number of α-repetitions determines the overall pitch range of the instrument. The number of physical buttons on any given keyboard lattice limits the total number of intervals; the choice of layout **L** determines the trade-off.

The vector position $\boldsymbol{v}_n$ of the nth note $n = \ldots, -2, -1, 0, 1, 2, \ldots$ with respect to the zeroth note in a swathe can be expressed as a function of α, β, $\boldsymbol{\psi}$, and $\boldsymbol{\omega}$ as

$$\boldsymbol{v}_n = n\boldsymbol{\omega} - \lfloor nz \rfloor \boldsymbol{\psi} \text{ where } z = \log_\alpha(\beta) \tag{8}$$

and $n \in \mathbb{Z}$. The slope and thickness (measured orthogonally to the slope) of the swathe are given by

$$m = \frac{\omega_y - \psi_y z}{\omega_x - \psi_x z} \text{ and } T = \frac{1}{\sqrt{(\omega_x - \psi_x z)^2 + (\omega_y - \psi_y z)^2}}. \tag{9}$$

The swathe is affected not just by the choice of generating vectors ($\boldsymbol{\psi}$ and $\boldsymbol{\omega}$) but also by the size of the generating intervals α and β via the variable z of (8). Clearly, the different VTRs of different regular temperaments have swathes with different thicknesses and slopes. Fig. 2 shows the swathes produced for generators with different z values ($z \approx \frac{7}{12}$, suitable for the meantone temperament; $z \approx \frac{2}{3}$ for the Magic temperament; $z \approx \frac{1}{4}$ for the Hanson temperament), when using the Wicki and Fokker layouts. The swathe thickness T gives a measure of the extent to which that swathe favours the number of α-reduced intervals compared to the number of α repetitions of those intervals. When T is large, the swathe is wide and so uses up more button-lattice space for α-reduced intervals and implies fewer repetitions. When T is small, the swathe is narrow and so leaves more lattice space for α-repetitions.

Given a specific button-lattice and tuning, the size of T (as determined by the layout) determines the highest cardinality α-reduced β-chain, and therefore MOS scale, that can be hosted. If all given button lattice shapes are approximated by a circular button-lattice, the maximum cardinality chain is proportional to T. When using the button-lattice illustrated in Fig. 2 with $\alpha = 2$ and a β-tuning of approximately 300 cents ($z = \frac{1}{4}$) the Fokker layout
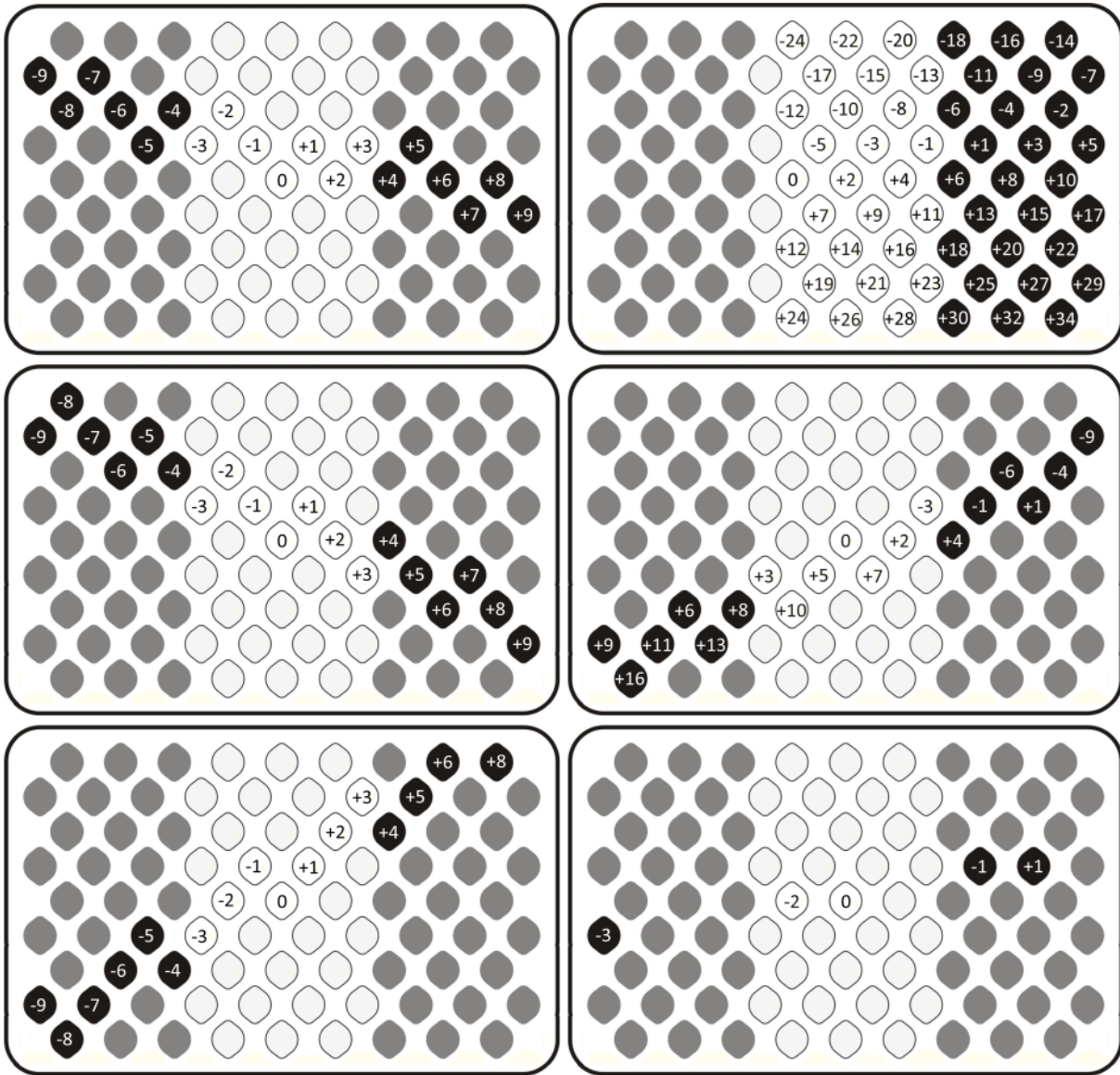
Figure 2. Swathes produced by the Wicki layout (left) and the Fokker layout (right) for $z = \frac{7}{12}$ (first row), $z = \frac{2}{3}$ (second row), $z = \frac{1}{4}$ (third row).

gives a maximum cardinality chain of only five notes, while Wicki manages a much more useful unbroken chain of 16 notes (buttons -9 to $+6$). On the other hand, at a β-tuning of approximately 700 cents ($z = \frac{7}{12}$), Fokker manages a remarkable 49-note unbroken chain length (buttons -18 to $+30$), while Wicki achieves 19 notes, so at this tuning Fokker can provide greater microtonal/modulatory resources, though over a smaller octave range.

For a given layout mapping $\mathbf{L}$, the T-curve (9) represents the change of swathe thickness T with respect to z. Thus T-curves indicate the range of tunings (and therefore temperaments) over which a given layout mapping provides usable swathes. T-curves for the Fokker and Wicki mappings are illustrated in Fig 3. Taking the derivative of (9) with respect to z and solving for $\frac{dT}{dz} = 0$ shows that the maximum swathe thickness occurs at

$$z^* = \frac{\psi_x \omega_x + \psi_y \omega_y}{\psi_x^2 + \psi_y^2}. \tag{10}$$

Substituting this into T (and using $|\det(\mathbf{L})| = 1$) shows that the maximum width of the swathe is

$$T_{\max} = T(z^*) = \sqrt{\psi_x^2 + \psi_y^2} = \|\boldsymbol{\psi}\| \text{ with slope } m_{\max} = -\frac{\psi_x}{\psi_y}.$$
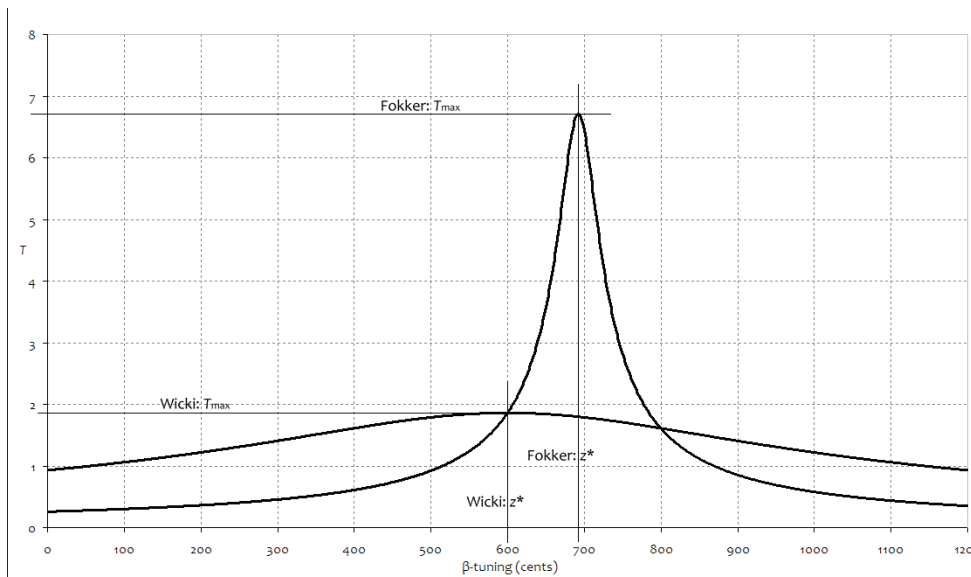
Milne, Sethares, Plamondon

Figure 3. T-curves for the Wicki and Fokker layout mappings.

Thus $T_{\max}$ is equal to the length of the basis vector ψ and the slope of the swathe at its widest point is $m_{\max}$.

The second derivative of T with respect to z shows the curvature of the T-curve at the point of maximal thickness. A low value indicates a wide gentle peak, a large value indicates a spiky peak:

$$T_{\text{curvature}} = \frac{d^2 T}{dz^2}\bigg|_{z=z^*} = -\frac{(\psi_x^2 + \psi_y^2)^{\frac{5}{2}}}{(\psi_x \omega_y - \psi_y \omega_x)^3} = \pm(\psi_x^2 + \psi_y^2)^{\frac{5}{2}} = \pm T_{\max}^5. \tag{11}$$

This shows that the higher the value of $T_{\max}$, the greater T's curvature, and so the more specific the layout mapping is to a smaller tuning range of generator intervals. A layout mapping can, therefore, only produce a wide swathe over a narrow tuning range; this is clearly illustrated by Figs. 3 and 4.

5   Examples of layouts and corresponding swathes

Fig. 4 shows the T-curves for all mappings that fit onto a hexagonal lattice with $T_{\max}$ no larger than eight. The highlighted T-curves show a selection of mappings, in order of height: $\mathbf{L}_1$, Wicki, $\mathbf{L}_2$, Chromatic Button Accordion (CBA-B and CBA-C have the same T-curve), and Fokker. The majority of layout mappings have a limited tuning range over which the swathe has a reasonable thickness. The Fokker layout mapping, for example, has a reasonable swathe width within a tuning range of approximately 520–860 cents; outside of this range the width drops below 1. Assuming $2 \mapsto \alpha$, this makes Fokker unsuitable for temperaments such as Hanson that require a small (or large) β. Specialised layouts such as Fokker, which have high values of $T_{\max}$, are only usable over a limited tuning range, but they are useful if there is a requirement for MOS scales of high cardinality (which provide a greater resource of octave-reduced intervals, at the expense of a restricted octave range) at specific tunings. Fokker is notable because its swathe thickness peaks within the meantone $2 \mapsto \alpha$ VTR, and so is particularly suitable for exploring microtonality within this tuning region. Using Fig. 4, it is straightforward to choose a layout mapping that is suitable for any given VTR or MOS scale.

A reasonable swathe thickness over a wide range of tunings is attained by members of a small class of mappings. This is exemplified by the Wicki layout, which has $T_{\max}$ at 600 cents and a swathe thickness that drops below one only in the first and last 90 cents of the tuning range. The $\mathbf{L}_1$ mapping is also usable over a wide tuning range, though its swathe width may be considered a little narrow within the important meantone VTR. With a good balance between octave-reduced intervals and overall pitch over the widest possible tuning range, Wicki is a particularly useful layout when a wide tuning range is required.

Another notable layout is $\mathbf{L}_2$, which has a good swathe thickness over the entire upper half of the tuning range.
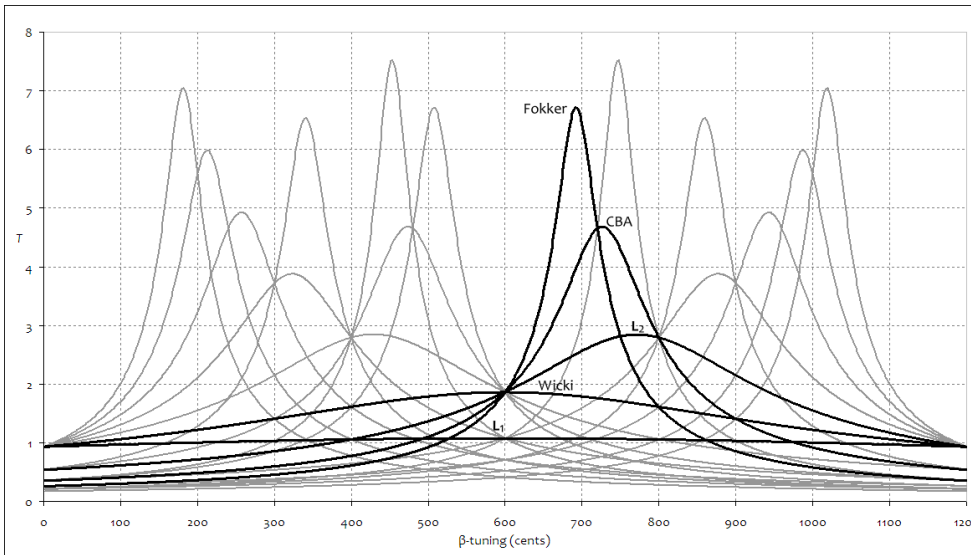
Figure 4. The T-curves of all possible mappings to a hexagonal lattice that have $T_{\max} < 8$. A selection of mappings from Table 2 are highlighted.

Right-multiplying this layout by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix} \right)$ reflects its z^* value about $z = \frac{1}{2}$, and so this has a good thickness over the lower half of the tuning range.

The relationship between MOS scales, temperaments, and layouts is illustrated in Fig. 5, which shows the maximum cardinality α-reduced β-chains for a selection of layouts hosted on an approximately hexagonal button-lattice, with a reduced basis of $\left(\begin{smallmatrix} 1 & 0.5 \\ 0 & 1 \end{smallmatrix} \right)$, and a diameter of ten buttons. The angular position around the circle corresponds to the β-tuning in cents (α is fixed at 1200 cents). The grey radial lines show the β-tunings that give n-edos and their cardinality n is indicated by the ring at which each line starts. The arcs between the n-edos represent the tuning ranges of all MOS scales, with cardinality indicated by the ring on which they are drawn. The bold radial lines indicate optimal tunings for various 5-limit regular temperaments, and the ring from which each line starts shows the lowest cardinality scale within which every scale note is a member of a full major or minor triad.

6   Isotones and pitch axes

Pitch axes are directions on a button field in which the pitch behaves in a simple way: an *isotone* is an axis in which the pitch remains fixed, the *orthogonal pitch axis* measures the shortest distance from a button to the isotone, and the *steepest pitch axis* points in the direction where the pitch increases (or decreases) most rapidly. Such axes can provide useful landmarks when playing in unfamiliar scales or tunings; they allow the fingering of different scales to be estimated (even without full knowledge of the layout being used) and allow easy visualisation of the pitch-density of chords and melodies. Pitch axes make a keyboard layout easier to learn and play because the pitch of every button can be estimated by sight and touch as well as hearing.

This section focuses on the $p - n = 2$ dimensional case, where a 2-D regular temperament can be characterized by two generators α and β. Let $g = (\log(\alpha), \log(\beta))$ be the tuning vector and let $g^{\perp}$ be the orthogonal complement (in Sect. 2 this was called the isotone hyperplane). Mapping the isotone via the keyboard mapping $\mathbf{L}$ gives

$$\mathbf{L}g^{\perp} = \begin{pmatrix} \psi_x & \omega_x \\ \psi_y & \omega_y \end{pmatrix} \begin{pmatrix} \log(\beta) \\ -\log(\alpha) \end{pmatrix} = \begin{pmatrix} -\omega_x \log(\alpha) + \psi_x \log(\beta) \\ -\omega_y \log(\alpha) + \psi_y \log(\beta) \end{pmatrix}$$

which points in the direction $\frac{-\omega_y \log(\alpha) + \psi_y \log(\beta)}{-\omega_x \log(\alpha) + \psi_x \log(\beta)} = \frac{\omega_y - \psi_y z}{\omega_x - \psi_x z}$ where z is defined in (8). Observe that this is equal to m of (9) and so the slope of the isotone is the same as the slope of the swathe. The isotone is a straight line across the button-lattice where the pitch remains unchanged. It may pass through the centres of some buttons: for example, when the tuning is 12-edo an isotone passes through the enharmonically equivalent F♭♭♭, E♭♭, D, C♯♯, B♯♯♯. Such cases represent particular values of α and β within the tuning continuum that have an additional comma. The shortest distance from a button to an isotone is monotonically related to its pitch, so a line drawn at right angles
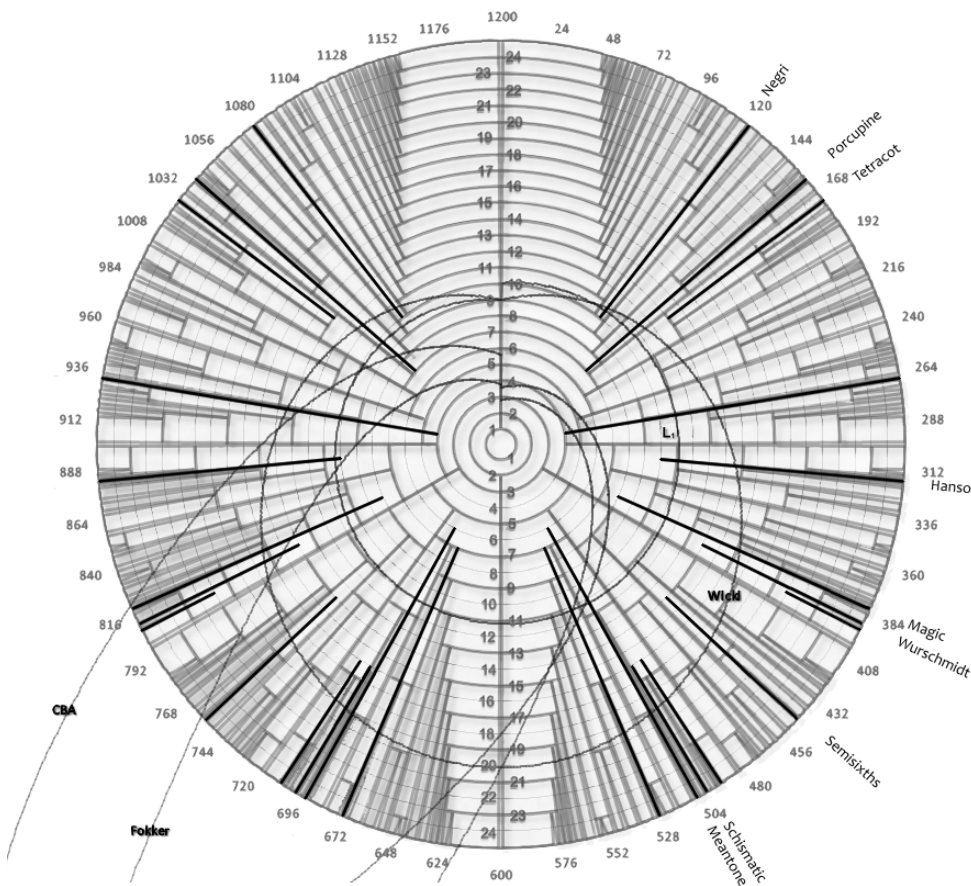
Figure 5. The maximum cardinality α-reduced β-chains for a selection of layouts on an approximately hexagonal button-lattice with a 10-button diameter.

to an isotone is the orthogonal pitch axis. Finally, the tuning vector is mapped to $\mathbf{L}g$, which points in the direction of the most rapid increase (or decrease, if negative) of the pitch.

It is convenient if the pitch axes are easily discernable. The most straightforward approach is to have (at least one) axis angled horizontally or vertically allowing the pitch of a button to be quickly determined according to its horizontal (or vertical) position on the button-lattice. For example, the Wilson layout ensures that $m = \infty$ when the tuning is 12-edo (i.e. at $z = \frac{7}{12}$). Because the angle of the isotones and pitch axes are dependent on the β-tuning, they can only be approximately horizontal or vertical over a limited range of tunings. Since the Wilson layout (like the Fokker and Bosanquet) is a specialised mapping that is usable over only a limited range of tunings, this does not present too much of an issue.

When a more generalist layout mapping is used, the broad range of tunings over which it can function inevitably means that the angle of the pitch axes vary substantially. For example, Fig. 6 shows the isotones for a selection of different β-values (α is fixed at 1200 cents) that run through the reference button D of a Thummer-style button-lattice using its default Wicki mapping (with $\mathrm{L_{Wic}}$ given in Table 2).

Observe how the isotone rotates clockwise about the reference button as the value of β increases. For example when $\beta = 600$ cents, the isotone is horizontal so the pitches of the buttons marked G♭, A♭, B♭, C, D, E, F♯, G♯, and A♯ are identical; when β is increased to 685 cents, the isotone rotates clockwise and the pitches of the buttons marked D♭, D, and D♯ become identical. Both these examples show how the conventional note names that are used in this figure become invalid outside the tuning range of diatonic recognisability ($2^{\frac{4}{7}} < \beta < 2^{\frac{3}{5}}$).

Somewhat paradoxically, the importance of obvious pitch axes is greater for generalist layout mappings than for specialist layout mappings. This is because the pitch order of the buttons changes as the tuning changes and, in the foreign landscape of an unfamiliar tuning, pitch axes provides a useful compass.

The following figures show how isotones may help to elucidate the intervallic structure of MOS scales, and the tunings at which the MOS scale passes between its different tuning phases. This is explained in detail for Fig. 7(a), which shows the six-note MOS scale generated when $\alpha^0 < \beta < \alpha^{\frac{1}{5}}$ (in the following examples α is fixed at 1200
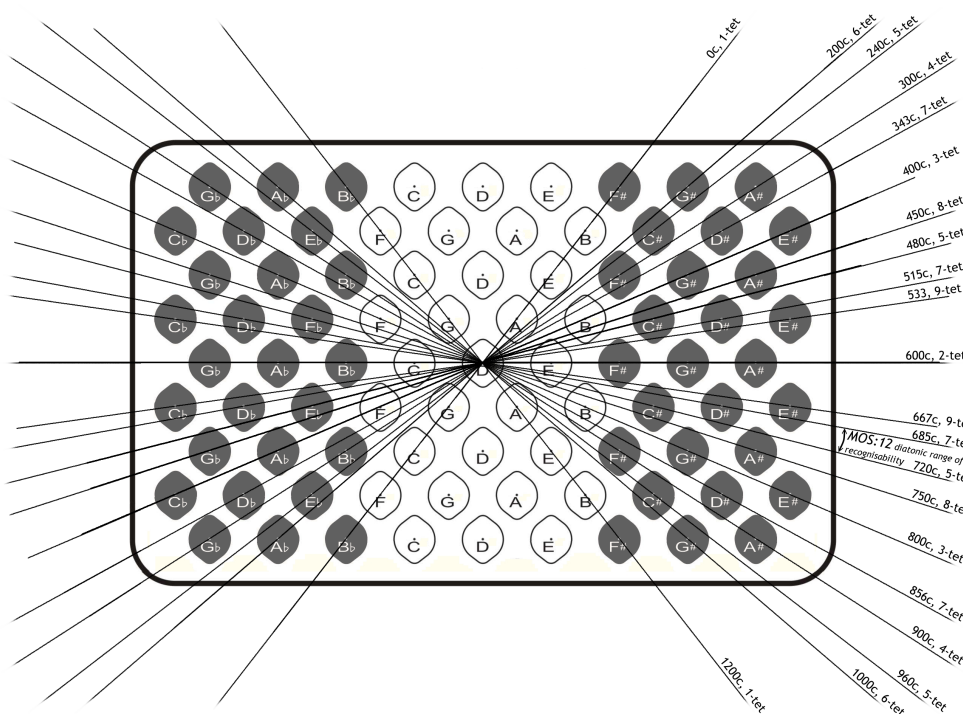
Figure 6.   A selection of isotones for the Wicki layout (the above note names are meaningful only within the range of recognisable diatonic tunings, but are provided here for reference).
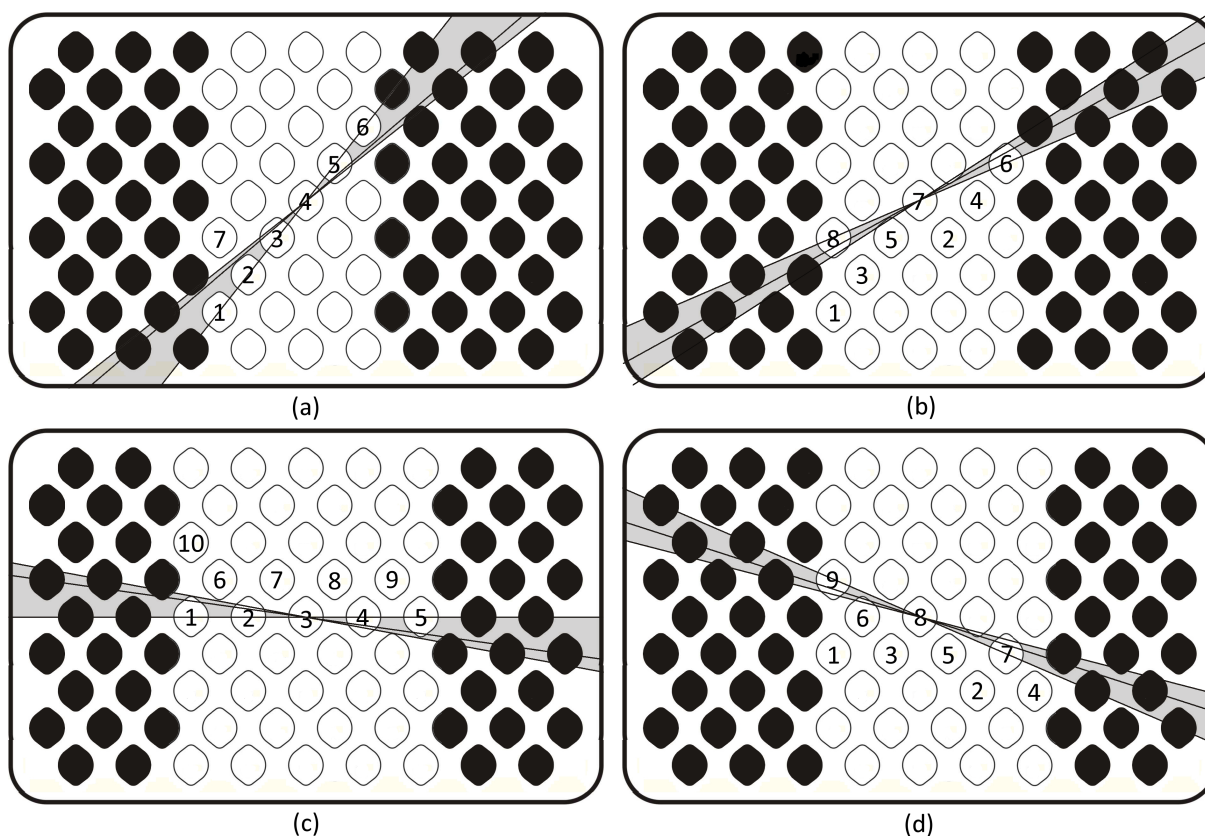

Figure 7.   (a) Six-note MOS scale with tuning range $0 < \beta < 240$ cents. (b) Seven-note MOS scale with tuning range $300 < \beta < 400$. (c) Nine-note MOS scale with tuning range $600 < \beta < 685$. (d) Eight-note MOS scale with tuning range $720 < \beta < 800$.

cents, so the β-tuning range is 0–240 cents). This tuning range is shaded and three isotones are shown: the boundary isotones at 0 cents and 240 cents, and the 6-edo isotone at 200 cents. This MOS scale has two types of scale step t_1 and t_2, as indicated on the layout by their differing shapes: t_1 is defined by $(0.5, 1)$ and is found between buttons 0, 1, 2, 3, 4, 5; t_2 is defined by $(-2.5, -3)$ and is found between buttons 5 and 6. The position of the isotones indicate the five different tuning phases of the MOS scale.

  (i) When β is 0 cents, $t_1 = 0$ cents which appears in the figure as buttons 0–5 all lie on the 0 cent isotone. This tuning has only octaves. It marks the non-inclusive boundary of the six-note MOS scale.

 (ii) As the β value increases, the isotone rotates clockwise so that the size on the monotonic pitch axis of t_1 increases and the size of t_2 decreases. This gives a six-note MOS scale containing five small steps t_1 and one large step t_2.

(iii) When $\beta = 200$ cents, the isotone rotates so that the sizes on the monotonic pitch axis of t_1 and t_2 are equal. This defines a scale with six equally sized steps, i.e. it is 6-edo.

(iv) For $200 < \beta < 240$ cents, the isotone continues to rotate and the size on the monotonic pitch axis of t_1 becomes larger than t_2. This gives a six-note MOS scale with five large steps t_1 and one small step t_2. This is the inverse of the scalic structure found in phase 2.

 (v) When $\beta = 240$ cents, buttons 5 and 6, which make t_2, occupy the same position on the monotonic pitch axis (they are parallel to the isotone) and t_2 has shrunk to 0 cents. This scale, therefore, contains just five equally sized intervals t_1, i.e. it is 5-edo. This is the non-inclusive boundary of the six-note MOS scale.

The same procedure (but with different numbers and positions of t_1 and t_2 intervals) occurs for any MOS scale and Fig. 7 provides several examples.

7   Harmonic and melodic considerations

Consider a *harmonic-melodic gamut* of intervals, the notes of which are typically played simultaneously (harmonic intervals) or consecutively (melodic intervals). Such intervals need to be spatially arranged on the keyboard so that they are close enough to be easily fingerable. Intervals outside of this gamut need not be spatially close. One way to specify the gamut is via the lowest cardinality MOS scale that contains a set of privileged intervals (3) augmented by the voice-leading intervals that connect them. For example, common practice suggests the use of the primary consonances of Example 3.1. Augmenting these intervals with the connecting intervals, the lowest cardinality MOS scale (generated by $\alpha = 2$ and β within a syntonic VTR) that contains all these intervals has 12 notes. Similarly, Magic temperament's harmonic-melodic gamut is contained within an MOS scale of 13 notes and Hanson temperament's by an MOS of 15 notes.

It is also necessary to consider an overall pitch range that needs to be easy to finger. In common practice, harmonic and melodic octaves occur frequently and this is reflected in the piano keyboard design where the octave is (approximately) the largest interval that a single hand can play. Generally it would seem reasonable for the range to be at least one interval of equivalence p_e.

Given a harmonic-melodic gamut defined by an α-reduced β-chain of n intervals (i.e. $n + 1$ notes), an overall pitch range of m intervals of equivalence, and a temperament basis where $p_e^{1/r} \mapsto \alpha$, the smallest parallelogram that contains these notes is bounded by the vectors $n(\boldsymbol{\omega} - z\boldsymbol{\psi})$ and $mr\boldsymbol{\psi}$. The magnitude of the diagonal of this parallelogram is $\|mr\boldsymbol{\psi} + n(\boldsymbol{\omega} - z\boldsymbol{\psi})\|$, which is a measure of the maximum possible finger-span required to play any interval within the gamut. Ideally the layout should be configured so that this finger-span is reasonably comfortable to play. The physical length of this diagonal is minimised when $mr\boldsymbol{\psi}$ and $n(\boldsymbol{\omega} - z\boldsymbol{\psi})$ are orthogonal and when $mr \|\boldsymbol{\psi}\| = \|n(\boldsymbol{\omega} - z\boldsymbol{\psi})\|$. Together, these imply

$$\|\boldsymbol{\omega}\| = \sqrt{\frac{(mr)^2 + n^2 z^2}{n^2}} \, \|\boldsymbol{\psi}\| \, .$$

8   Conclusions

A higher-dimensional tuning system such as a JI can be mapped to a lower-dimensional tuning system by tempering its intervals. The resulting temperament can be characterised by those intervals (commas) that are tempered to unison; the temperament's basis, and mappings to it, can be obtained from the null space of those commas. Any specific value of the basis elements represents a point on a tuning continuum of all possible values; the VTR represents the range of values for which all privileged intervals maintain their scalic ordering.

A linear and invertible mapping from a temperament basis to a button-lattice gives: a layout with invariant fingering for all intervals across all keys, invariant fingering for all privileged intervals across the VTR, and pitch axes enabling easy visualisation of the relative pitches of buttons. A swathe and a harmonic-melodic gamut are mathematically defined measures of musically important features, and enable concrete comparisons between different button geometries. A number of different equal and non-equal, octave and non-octave based temperaments demonstrate the generality of the tuning-theoretic ideas; a number of different layouts (such as those defined in Table 2) demonstrate that these results can be applied directly to any two-dimensional button-lattice.

Reference

[1] Milne, A., Sethares, W.A. and Plamondon, J., 2007, Isomorphic Controllers and Dynamic Tuning: Invariant Fingering Over a Tuning Continuum, *Computer Music Journal*, vol. 31, no. 4, Winter.

[2] Sethares, W.A., Milne, A. and Plamondon, J., 2007, Consistent fingerings for a continuum of syntonic commas. Presentation given at American Mathematical Society 2007, New Orleans, LA, 6 Jan.

[3] Various, Alternate Tunings Mailing List. Available online at: http://launch.groups.yahoo.com/group/tuning/

[4] Smith, G.W., Xenharmony. Available online at: bahamas.eshockhost.com/~xenharmo/ (accessed October, 2006).

[5] Erlich, P., 2006, A Middle Path Between Just Intonation and the Equal Temperaments, Part 1, *Xenharmonikon* vol. 18, pp. 159-199.

[6] Sethares, W.A., 1997, *Tuning, Timbre, Spectrum, Scale*. (Springer-Verlag. Second edition 2004).

[7] Cohen, E.A., 1984, Some effects of inharmonic partials on interval perception, *Music Perception* Vol. 1, No. 3, 323-349.

[8] Wilson, E., 1975, Letter to Chalmers Pertaining to Moments Of Symmetry/Tanabe Cycle. Available online at: www.anaphoria.com/mos.PDF (accessed 26 March 2007).

[9] Carey, N. and Clampitt, D., 1989, Aspects of Well-Formed Scales. *Music Theory Spectrum*, 11, 187-206.

[10] Noll, T., 2005, Facts and Counterfacts: Mathematical Contributions to Music-theoretical Knowledge, In: Sebastian Bab, et. al. (eds.) *Models and Human Reasoning* (Berlin: W&T Verlag)

[11] Carey, N., 2006, Coherence and sameness in well-formed and pairwise well-formed scales. *Journal of Mathematics and Music*, vol. 1, no. 2, July, pp. 79-98.

[12] Blackwood, 1985, *The Structure of Recognizable Diatonic Tunings* (New Jersey, Princeton University Press).

[13] Tonalsoft, Encyclopedia of Microtonal Music Theory. Available online at: tonalsoft.com/enc/e/equal-temperament.aspx (accessed 26 March 2007).

[14] Regev, O., 2004, Lecture 1 Introduction. Available online at: www.cs.tau.ac.il/~odedr/teaching/lattices_fall_2004/ln/introduction.pdf (accessed 26 March 2007).

[15] Weisstein, E.W., Circle Packing. From *MathWorld*–A Wolfram Web Resource. Available from mathworld.wolfram.com/CirclePacking.html (accessed 26 March 2007).

[16] Thumtronics Ltd., Thumtronics: The New shape of Music. Available online at: www.thummer.com (accessed 26 March 2007).

[17] Starr Labs, Home Page. Avialable online at: www.starrlabs.com (accessed 26 March 2007).

[18] C-Thru Music Ltd., Home Page. Available online at: www.c-thru-music.com (accessed 26 March 2007).

[19] H-Pi Instruments, Tonal Plexus. Available online at: www.h-pi.com/TPX28intro.html (accessed 26 March 2007).

[20] Fokker, A., 1955, Equal Temperament and the Thirty-one-keyed Organ. Available online at: www.xs4all.nl/ huygensf/doc/fokkerorg.html (accessed 26 March 2007).

[21] Wicki, K. 1896, Tastatur für Musikinstrumente. Swiss patent Nr. 13329, in Neumatt (Münster, Luzern, Schweiz) 30 October 1896.

[22] Helmholtz, H.L.F., 1877, *On the Sensations of Tones*. Trans. Ellis, A.J.. Reprint by Dover in 1954 (New York: Dover)

[23] Wilson, E., 1974, Bosanquet – A Bridge – A Doorway to Dialog. Available online at www.anaphoria.com/xen2.PDF (accessed 26 March 2007).