Invariance of Controller Fingerings across a Continuum of Tunings

Andrew Milne[**], William Sethares[††], and James Plamondon[‡‡]

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Introduction

On the standard piano-style keyboard, intervals and chords have different shapes in each key. For example, the geometric pattern of the major third C–E is different from the geometric pattern of the major third D–F♯. Similarly, the major scale is fingered differently in each of the twelve keys (in this usage, fingerings are specified without regard to which digits of the hand press which keys). Other playing surfaces, such as the keyboards of [Bosanquet, 1877] and [Wicki, 1896], have the property that each interval, chord, and scale type have the same geometric shape in every key. Such keyboards are said to be *transpositionally invariant* [Keisler, 1988].

There are many possible ways to tune musical intervals and scales, and the introduction of computer and software synthesizers makes it possible to realize any sound in any tuning [Carlos, 1987]. Typically, however, keyboard controllers are designed to play in a single tuning, such as the familiar 12-tone equal temperament (12-TET) which divides the octave into twelve perceptually equal pieces. Is it possible to create a keyboard surface that is capable of supporting many possible tunings? Is it possible to do so in a way that analogous musical intervals are fingered the same throughout the various tunings, so that (for example) the 12-TET fifth is fingered the same as the Just fifth and as the 17-TET fifth? (Just tunings

[**]Corresponding author. Email: andymilne@tonalcentre.org

[††]Department of Electrical and Computer Engineering, University of Wisconsin-Madison, Madison, WI 53706 USA. Email: sethares@ece.wisc.edu

[‡‡]Thumtronics, Austin, TX. Email: research@thumtronics.com

are those with intervals defined by ratios of small integers; for example the Just fifth is the ratio 3/2 and the Just major third is 5/4.)

This paper answers this question by presenting a single example of a *tuning continuum* (a parameterized family of tunings where each specific tuning corresponds to a particular value of the parameter) which exhibits *tuning invariance* (where all intervals and chords within a specified set have the same geometric shape in all of the tunings of the continuum).

Such an isomorphic, invariantly-fingered button-field has three advantages. First, having a single set of fingerings within and across all keys of any given tuning makes it easier to visualize the underlying structure of the music. Second, having this same single set of interval shapes across the tuning continuum makes it easier for musicians to explore the use of alternative tunings such as the various meantones, Pythagorean, 17-TET, and beyond. Third, assigning the continuous parameter to a control interface enables a unique form of expression: dynamically (re)tuning all sounded notes in real time where the scalar function of the notes remains the same even as the tuning changes.

The Thummer keyboard [Thumtronics, 2007], shown in Figure 1, is used to concretely demonstrate the layout of the pitches and notes on a practical keyboard surface. This article focuses on the mathematical principles behind the ideas of transpositional and tuning invariance, though other keyboard layouts such as those of Wicki (1896) or Bosanquet (1876) could also have been used.

Figure 1: Thumtronics forthcoming USB-MIDI controller, the Thummer, contains two keyboards, each with 57 pressure-sensitive buttons. A variety of controllers include two thumb-operated joysticks and optional internal motion sensors. One side of the Thummer layout is used to concretely demonstrate the possibility of tuning and transpositional invariance.

There are several technical, musical, and perceptual questions that must be addressed in order to realize a keyboard that is both transpositionally and tuning invariant. First, there must be a range of tunings over which pitch intervals, and therefore their fingerings, remain in some sense "the same." This requires that differently tuned intervals be identified as serving the same role; for instance, that the 12-TET fifth be identified with the Just fifth and the 19-TET fifth. Said differently, tuning invariance requires that there be a number of distinguishable intervals by which the invariance can be measured, because in order to say that two numerically different intervals are both fifths, it is necessary to identify a perfect fifth as an interval distinguishable from a major third or a perfect fourth, or a diminished fifth, etc. This issue of the identity of musical intervals is discussed in detail in the next section by contrasting *rational* (simultaneous) and *ordinal* (successive) methods of interval identification.

Second, there must be a standard to which intervals can be compared. The section "Tuning Systems and Temperaments" distinguishes *regular tunings* from *regular temperaments*. Regular tunings are defined to be those that can be generated from a finite number of intervals; for example, Pythagorean tuning is generated from the octave and the just fifth. Regular temperaments presume that there is an underlying Just Intonation (JI) tuning system which is mapped to a regular tuning in a structured way so that certain intervals retain their identity. This tempering is defined mathematically using a comma (we choose the syntonic comma, an interval of 81/80, for our example) which parameterizes the regular temperament with a pair of generators; those tunings which can be reached by this parameterization form the tuning continuum of the syntonic comma. Combining this with the sameness criteria for intervals defines the range of the tunings over which chords and scales retain consistent shapes.

Third, it is necessary to map the regular temperament to a keyboard button field. The layout-mappings of the section "Button-lattices and Layouts" translate the generating intervals of the tuning to the keyboard surface. It is shown that transpositional invariance is identical to linearity of the layout-mapping. Successive sections then provide examples of keyboard layouts that are (and others that are not) invariant in both transposition and tuning. The syntonic continuum, pictured in Figure 2, provides the primary example of this paper. It encompasses 7-TET, moves continuously through various mean-tone tunings, 12-TET, 17-TET (and many others) and ends at 5-TET, retaining fingering invariance throughout. Finally, a musical example illustrates static snapshots of the dynamic retuning process.

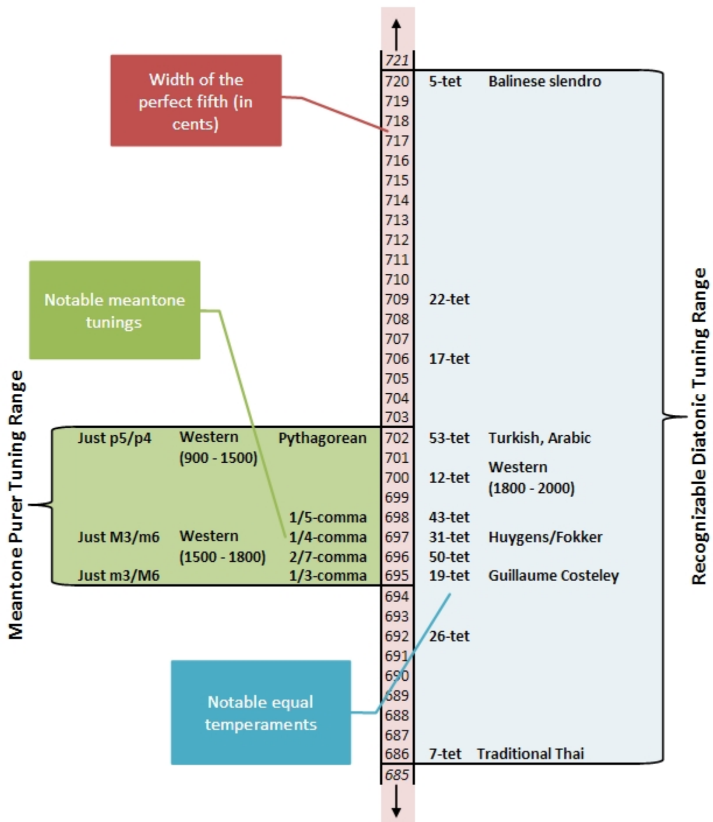

Figure 2: The complete syntonic tuning continuum with several notable meantone, diatonic, and n-TET tunings highlighted.

Intervals

This section investigates how intervals are identified and distinguished, and discusses criteria by which two numerically different intervals may be said to play (or not to play) analogous roles in different tuning systems. Intervals may be identified, and therefore discriminated in at least two ways: the rational (simultaneous) and the ordinal (successive).

Rational Identification

This mode of identification primarily occurs for harmonic intervals, those formed from simultaneously sounded notes. It presumes that the Just intervals, which are perceived as consonant, act as perceptual and cognitive landmarks (a template) against which sounded intervals can be mentally compared and identified. This is reasonably uncontroversial for harmonic orchestral instruments. For inharmonic computer–generated sounds, inharmonic bells, and non-Western instruments such as the metallophones of the Indonesian gamelan, the sensory consonances may occur at different intervals [Sethares, 2004] and so other templates may be more appropriate.

For harmonic sounds, when a sounded interval is tuned close to a low integer ratio JI interval (such as 3/2 or 5/4) it may be heard as a representation of it. In other words, the sounded interval is an indexical signifier (Chandler 2001) of the Just ratio it approximates. For example, when an interval is tuned to 702 cents (the closest integer value to 3/2), it can be heard as a representation of the 3/2 Just fifth. As the interval's tuning is moved away from 702 cents, it gradually moves to a state where it is likely to be heard as an imperfect representation of 3/2 (it will sound more or less "out-of-tune"). As the tuning is moved still further from 702 cents, the perceived interval will eventually no longer represent 3/2 (not even an out-of-tune 3/2). At this point the Just interval is no longer signified by the sounded interval. When an interval is not Just, but is within its range of rational identification, it is called a tempered interval.

Ordinal Identification

This mode of identification primarily occurs for melodic intervals (formed from successively sounded notes). It presumes that when an interval is played as part of a conventionalized (Chandler 2001) scale, it is identified by the number of scale notes (or steps) that it spans. For example, [Wilson, 1975] writes that "our perception of Fourth-ness is not just acoustic, i.e. 4/3 determined, it is melodic and/or rhythmic influenced to a high degree." This is also an indexical signification, but one that requires the presence of a scalic background (context) to give meaning to "second-ness," "third-ness," "fourth-ness," etc.

Interval Identification

The two modes of identification overlap: the rational mode plays a part in the identification of melodic intervals; the ordinal in the identification of harmonic intervals. For example, a melodic interval of approximately 3/2 will be heard as "in tune" or "out of tune" according to its proximity to this Just interval; similarly, a harmonic interval approximating 3/2 will, in a traditional diatonic context, be heard as spanning five notes of the scale.

Furthermore, in real-world music it is not always possible to make a strict distinction between harmonic intervals and melodic intervals. An arpeggio is at least partially a harmonic structure. The bass note of a "stride bass" pattern, which is sounded only for the first and third beats of a bar, cognitively grounds the rest of the bar. Counterpoint, which is the interweaving of many melodies into a coherent harmonic structure, blurs the distinction between melody and harmony.

Thus the rational and ordinal modes of identification are intertwined. Indeed, in Western tonal music, there is a consistent linkage between the number of steps an interval contains and its harmonic ratio. For example, the interval that spans three scale notes (i.e., the third) is commonly an interval which is close to 5/4 or 6/5 (hence the names major third and minor third). Similarly, the interval that spans four scale notes (i.e., the fourth) is typically an interval which is close to 4/3. Furthermore, where these links differ, the interval often has a tonally dissonant function that requires resolution to a more stable interval (for example, an augmented second might resolve to a perfect fourth, or a diminished fifth may resolve to a major third).

In conventionalized musical systems such as Western tonal music, where particular step sizes and harmonic ratios are consistently associated, ordinality can (by association) symbolically signify ratio, and ratio can (by association) symbolically signify ordinality. This means that, within the conventionalized context, the tuning ratio over which an interval can still signify a given Just ratio may be wider than expected if it were judged by analyzing harmonies isolated from the musical context. In common practice, for example, the conventional association of "5/4-ness" and "third-ness" means that a melodic third can be tuned very wide but still signify 5/4 – the ultra-sharp supra-Pythagorean major thirds ($>$ 408 cents) may be sometimes used by string players for expressive intonation [Sundberg, 1989]. Though this may be harmonically uncomfortable, it is likely to retain its identification with a Just quarter-comma meantone thirds of 386 cents. Thus the tuning range over which a given interval can be identified and therefore discriminated from other intervals has "fuzzy" boundaries that are context dependent.

The previous discussion suggests that it may be advantageous to define the tuning range over which an interval preserves both its rational and ordinal identity as that interval's tuning range of invariant identification. Since there are two modes by which an interval can be identified, there are two relevant tuning ranges to be considered, which themselves depend on a listener's innate abilities, experiences, and training. Musical context is also important; the tuning range of invariant identity may also change based on the spectrum and/or timbre of the sounds [Sethares, 2004]. For these reasons, the final judgment as to the specific tuning boundaries may best lie in the hands of the artist and not the theoretician.

Nonetheless, in certain situations, ordinal identification can be precisely bounded. Given a scale in a regular tuning system (as defined below), the size of every interval in the scale is determined by the values of the generating intervals. This implies that the range over which intervals can be retuned but still maintain their ordinal identity is limited by the points at which one or more of that scale's steps shrink in size to zero (or equivalently, the points at which two of the scale steps "cross"). This range is shown concretely in Figure 2 for the regular temperament defined by the syntonic comma.

Tuning Systems and Temperaments

A *tuning system* is defined here to be a collection of precisely tuned musical intervals. There are many ways in which the intervals may be chosen: a "boutique" tuning system might have all of its intervals chosen arbitrarily, another tuning system might be generated by a predefined mathematical procedure [McLaren, 1991]. One concrete class of tuning systems are the *regular tunings* in which all of the intervals are generated multiplicatively from a finite number of generating intervals. For example, 3-limit Just Intonation has two generators $T_1 = 2$ and $T_2 = 3$, and consists of all products of the form $T_1^i T_2^j = 2^i 3^j$ where i and j are integers. Thus the intervals of 3-limit JI can all be found in a series of stacked, Just perfect fifths, allowing for octave equivalence. In general, a regular tuning is characterized by n generators T_1 to T_n and consists of all intervals $T_1^{i_1} T_2^{i_2} \cdots T_n^{i_n}$ where the $i_1, i_2, \ldots, i_n$ are integer-valued exponents.

Altering the tuning of a generator affects the tuning of the system in a predictable way. For example, the fifth in 3-limit JI is $T_1^{-1}T_2$ (i.e., $2^{-1}3$). If a (non-JI) regular tuning is created by changing the value of T_2, the value of the fifth changes in a patterned way. Assigning the magnitude of one or more of the generating intervals to a control interface provides a convenient means to "navigate" the tuning continuum. Particular values within this continuum may produce some intervals that approximate JI intervals and so are rationally identifiable.

Such a structured mapping from JI to a regular tuning is called a *regular temperament*. We show below that such mappings must be linear, though they need not be invertible (i.e., they need not be one-to-one). Moreover, these mappings can be characterized by small JI intervals called commas that are tempered to unison [Smith, 2006].

To be concrete, two or more intervals $a_1, a_2, \ldots, a_n$, are said to be multiplicatively dependent if there are integers $z_1, z_2, \ldots, z_n$, not all zero, such that $a_1^{z_1} a_2^{z_2} \cdots a_n^{z_n} = 1$. If there are no such z_i, then the a_i are said to be *multiplicatively independent*. The *rank* of a tuning system is the number of multiplicatively independent intervals needed to generate it. A regular temperament typically has lower rank than the JI system that is mapped to it (i.e. the mapping is non-invertible). When the temperament mapping loses rank, all intervals can no longer be Just. But, as long as there is a range of generator values over which the intervals are correctly identified, the regular temperament can be considered to be rationally valid.

This is analogous to the way a projection of the three-dimensional surface of a globe to a two-dimensional map inevitably distorts distance, area, and angle; but so long as the countries have identifiable shapes, the projection can be considered valid. Different map projections result in different distortions, and some map projections are more or less suitable to specific purposes. Some projections (such as the Mercator Projection) have the virtue of wide familiarity; so it is also with temperament-mappings (such as those that lead to 12-TET).

For example, 3-limit JI can be temperament-mapped to a one dimensional system using an appropriate comma $T_1^a T_2^b = 1$, where a and b are integers. One notable tempering retains the octave $T_1 = 2$ and tempers T_2 to $2^{19/12} \approx 3$. Choosing $a = 19$ and $b = -12$ results in the familiar 12-TET. This comma can be interpreted musically by saying that in this temperament, 19 octaves minus twelve equal-tempered perfect twelfths equals a unison.

A second example is in 5-limit JI, which consists of all intervals of the form $2^i 3^j 5^k$ where i, j, and k

are integers. This can be reduced to a two-dimensional regular temperament by choosing T_1 (typically near 2), T_2 (near 3), and T_3 (near 5) so that

$$T_1^a T_2^b T_3^c = 1, \tag{1}$$

where a, b, and c are specified non-zero integers. Equation (1) defines a comma. The well known syntonic comma is the special case where $a = -4$, $b = 4$, and $c = -1$; tempering the generators so that Equation (1) holds produces various meantone temperaments [Meantone Citation]. The comma can be solved for one of its terms as ($T_3 = T_1^{a/c} T_2^{b/c}$) and a typical interval of the regular temperament can be written in terms of two generating intervals $\alpha = T_1^{\text{GCD}(a,c)/|c|}$ and $\beta = T_2^{\text{GCD}(b,c)/|c|}$ where $\text{GCD}(n, m)$ is the greatest common divisor of n and m. Thus $T_1 = \alpha^{|c|/\text{GCD}(a,c)}$, $T_2 = \beta^{|c|/\text{GCD}(b,c)}$, $T_3 = \alpha^{a\text{SIGN}(c)/|\text{GCD}(a,c)|} \beta^{b/\text{SIGN}(c)|\text{GCD}(b,c)|}$, and a typical interval $T_1^i T_2^j T_3^k$ of the JI is temperament-mapped to $\alpha^{(ciak)/|\text{GCD}(a,c)|} \beta^{(cjbk)/|\text{GCD}(b,c)|}$. In matrix notation, the vector $(i, j, k)'$ is temperament-mapped by

$$T = \begin{pmatrix} \frac{|c|}{\text{GCD}(a,c)} & 0 & \frac{-a\text{SIGN}(c)}{\text{GCD}(a,c)} \\ 0 & \frac{|c|}{\text{GCD}(b,c)} & \frac{-b\text{SIGN}(c)}{\text{GCD}(b,c)} \end{pmatrix}$$

to $\left(\frac{i|c| - ak\text{SIGN}(c))}{\text{GCD}(a,c)}, \frac{j|c| - bk\text{SIGN}(c))}{\text{GCD}(b,c)} \right)'$ where the prime $'$ represents the transpose. Thus the temperament-mapping is linear and (typically) loses rank. For the syntonic comma, $T = \left(\begin{smallmatrix} 1 & 0 & -4 \\ 0 & 1 & 4 \end{smallmatrix} \right)$ and any note $(i, j, k)'$ is mapped to $(i - 4k, j + 4k)'$. For example, a Just major third $5/4 = 2^{-2} 3^0 5^1$ is represented by the vector $(-2, 0, 1)'$. This is mapped by T to $(-6, 4)'$, which represents the tempered interval $\alpha^{-6} \beta^4$. If α were tempered to 2 (no change in the octave) and β were tempered to $2^{30/19}$ (the 19-TET fifth) then $\alpha^{-6} \beta^4 = 2^{-6} 2^{(120/19)} \approx 1.24469 \approx 5/4$ is the 19-TET approximation to the major third.

Scales

A *scale* is here defined to be a subset of a tuning system used for a specific musical purpose. When using a rank-two tuning system with generators α and β, a scale can be simply constructed by stacking integer powers of β and then reducing (dividing or multiplying by integer powers of α) so that every term lies between 1 and α. This is called an α-reduced β-chain, and it produces a scale that repeats at intervals of α; $\alpha = 2$, representing repetition at the octave, is the most common value (one says "octave-reduced"). Any arbitrary segment of an α-reduced β-chain can be used to form a scale, and the number of notes it contains is called its cardinality. For a given tuning of α and β, certain cardinalities are moment of symmetry (MOS) scales [Wilson, 1975] (also known as well-formed scales [Carey, 1989]), which have a number of musically advantageous properties: they are distributionally even, which means that the scale has two step sizes that are distributed as evenly as possible [Clough, 1999]; they have constant structure, which means that every given interval always spans the same number of notes [Grady, 1999]. Two familiar MOS scales generated by $\alpha = 2$ and $\beta \approx 3/2$ are the five-note pentatonic scale (e.g. D, E, G, A, C) and the seven-note diatonic scale (e.g. C, D, E, F, G, A, B); other β-values generate MOS scales with quite different intervallic structures. These provide fertile resources for non-standard microtonal scales.

Every MOS scale with specified generating intervals and cardinality has a valid tuning range over which it can exist; beyond this range, notes need to be added to or removed from the scale to regain distributional evenness and constant structure. The boundaries of this tuning range mark the tuning points at which some of the scale's steps shrink to unison, and therefore the tuning points at which the ordinal identity of intervals in that scale changes. A MOS scale generated by α and β with cardinality c has a valid tuning range of $\alpha^{p/q} < \beta < \alpha^{r/s}$ where p/q and r/s are adjacent members of a Farey sequence of order $c - 1$ (a Farey sequence of order n is the set of irreducible fractions between 0 and 1, with denominators less than n, arranged in increasing order).

The 12-note MOS scale generated by $\alpha = 2$ and $\beta \approx 3/2$, which is the chromatic scale that provides the background for common practice music, has a valid tuning range of $2^{4/7} < \beta < 2^{3/5}$ ($4/7$ and $3/5$ being adjacent members of the Farey sequence of order 11). Observe the familiar seven note diatonic scale as the tuning traverses this range - starting with a generating interval β tuned to the 12-TET fifth $2^{7/12}$, the scale consists of five large steps (whole tones) and two small steps (half tones). As β is increased, the whole tones grow larger and the half tones grow smaller. When β reaches $2^{3/5}$, the semitones disappear completely; above $2^{3/5}$, there is no seven note MOS scale available because the seven note scale contains three different step sizes. On the other hand, as β is decreased below $2^{7/12}$, the tones get smaller and the semitones get larger. When β reaches $2^{4/7}$, the tones and semitones become the same size; below $2^{4/7}$, the MOS scale's internal structure changes so that it has five small intervals and two large intervals, a scale structure that is the inverse of the familiar diatonic scale. These changing step sizes are illustrated in Figure 3. This tuning range is equivalent to Blackwood's range of recognizability for diatonic tunings [Blackwood, 1985]. The valid tuning ranges for MOS scales are, therefore, a generalization of the concept of range of recognizability beyond the familiar diatonic/chromatic context.

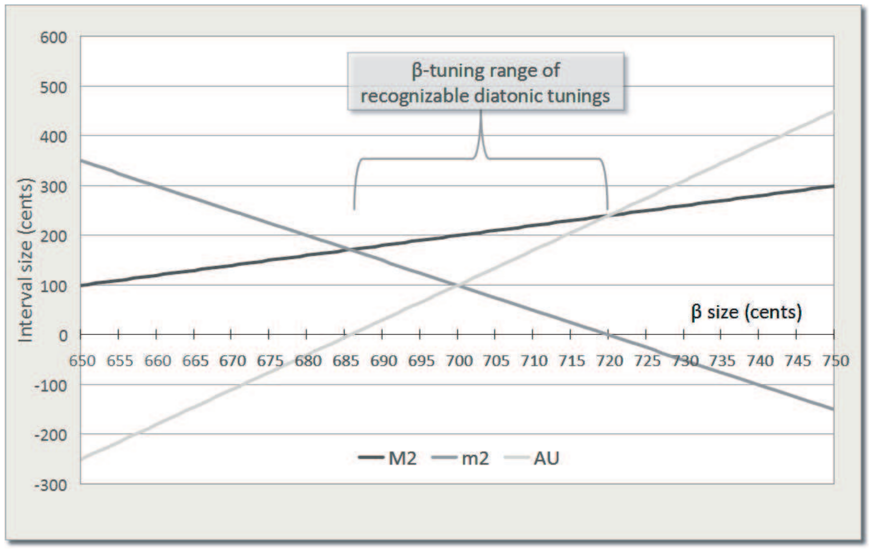

Figure 3: Size of major second (M2), minor second (m2), and augmented unison (AU), over a range of β generator tunings

When intervals are played as a part of a conventionalized or aesthetically consistent scale, they can be ordinally identified by the number of notes or intervals they span. Due to their distributional evenness and constant structure, MOS scales are likely to be heard as intrinsically aesthetically consistent, so the valid tuning range of any given MOS scale is equivalent to the range within which all of its intervals are ordinally invariant. Furthermore, these ranges hold for many musically useful alterations of MOS scales such as the harmonic minor scale (e.g. A, B, C, D, E, F, G♯). Thus the tuning range of ordinal validity for an n-note MOS scale derived from a rank-two temperament is that (two-dimensional) range of generating interval tunings equivalent to the range of validity of that MOS scale.

Button Lattices and Layouts

Instruments capable of playing a number of discrete pitches may use many buttons. For an instrument to have transpositional invariance, it is necessary that the buttons be arranged as a lattice [Wickipedia, 2006a]. If the lattice has one dimension, it may be called a button-row; if it has two-dimensions it may be called a button-field. A *layout* is here defined as the physical embodiment of an invertible, but not necessarily linear, mapping from a regular temperament to an integer-valued button lattice. In the same way that a regular temperament has a finite number of generating intervals (e.g. α and β) that generate all its intervals, a lattice can be represented with a finite number of basis or generating vectors that span its surface. The logical means to layout-map from a temperament to a button lattice is to map the temperament's generating intervals to the lattice's generating vectors.

Let $\mathbf{L} : \mathbb{Z}^n \to \mathbb{Z}^n$ map from the n generating intervals of the temperament to the n-dimensional button field. For example, with $n = 2$, the temperament contains two generating intervals α and β. Any interval of the temperament can be expressed as a two-vector $(j, k)'$ representing the interval $\alpha^j \beta^k$. The standard basis for the temperament consists of the generating intervals $(1, 0)'$ and $(0, 1)'$, and the layout-mapping $\mathbf{L}$ is the 2x2 matrix

$$\mathbf{L} = \begin{pmatrix} \psi_x & \omega_x \\ \psi_y & \omega_y \end{pmatrix}$$

which transforms the temperament's generating intervals into the lattice's generating vectors $\psi = (\psi_x, \psi_y)'$ and $\omega = (\omega_x, \omega_y)'$. The elements of $\mathbf{L}$ must be integers (or else some intervals will be layout-mapped to locations without buttons), $\mathbf{L}$ must be invertible (that is, det($\mathbf{L}$) $\neq 0$, or else either some buttons would have no assigned note or some notes would have no corresponding button) and det($\mathbf{L}$) must be ± 1 (or else the inverse will not be integer valued). This mapping provides the mathematical setting for two results:

Theorem 1 *If the layout-mapping* $\mathbf{L}$ *is linear, the layout is transpositionally invariant.*

Moreover, the converse holds as well:

Theorem 2 *If the keyboard layout is transpositionally invariant, the layout-mapping* $\mathbf{L}$ *must be linear.*

Theorem 2 justifies the use of linear (matrix) notation for the layout-mapping. Proof of the theorems is given in Appendices A and B. The theorems presume an infinitely sized button array; physical keyboards will necessarily have finite size and the linearity will be violated at the edges. The finite size also limits the number of octaves that can be realized. There are many possible layout matrices $\mathbf{L}$ and several concrete examples are given in the next section. The choice of $\mathbf{L}$ impacts the ease with which particular scales and chords can be fingered, and the number of octaves that can fit on a keyboard of a given geometry.

As successive notes in an α-reduced β-chain are mapped to a button-field, they cut a swathe across the lattice. The size of the swathe determines the microtonal and modulatory capabilities of the instrument; the number of α-repetitions determines the overall pitch range of the instrument while the number of physical buttons on the keyboard limits the total number of intervals. The choice of layout $\mathbf{L}$ determines the trade-offs. Certain keyboards, such as that of [Fokker, 1955], vary greatly as the tuning is changed; others, such as the Wicki, are more consistent across a range of tunings. Quantifying such observations is an important area for future investigation.

Since a layout-mapping is invertible, by definition, it can only be linear if it does not lose rank; such a layout-mapping is called *isomorphic*. A layout is, therefore, isomorphic across the valid tuning range of a given temperament if it uses all of the same generating intervals as that temperament. Translating into words, the theorems show that given a regular temperament, transpositional invariance requires an isomorphic layout-mapping to a button lattice. Moreover, given a regular temperament and a MOS scalic background, tuning invariance requires that the tunings of the generating intervals (as embodied in the layout) remain within that temperament's tuning range for invariant identification. This means that transpositional invariance on a button-row is only possible for a rank-one (i.e. equal) temperament. For a rank-two temperament like meantone, a button-field of at least two dimensions is required. The following examples illustrate by demonstrating concrete designs that are invariant in both transposition and tuning.

Examples of Layout-mappings

The examples in this section use meantone, 12-TET, and 19-TET regular temperaments, and assume a twelve note MOS scalic background – design choices that are not incompatible with tonal music. Two types of layouts are shown. One dimensional button-rows provide the simplest setting; drawings such as Figures 4–6 should be interpreted as consisting of a single row of identical buttons. For two dimensional button-fields, there are several ways in which the buttons can be arranged: in a rectangular grid, in offset rows (like brickwork), in a hexagonal grid, or as a tiling of parallelograms. The [Wicki, 1896] layout-mapping onto a Thummer-style hexagonal button lattice [Thumtronics, 2007] provides our primary two-dimensional example, though obviously other keyboard geometries could be used.

Layout-mapping from a Rank-one Temperament to a Button-row

The familiar 12-TET has a single generating interval α of $2^{1/12}$ (100 cents) and tempers out the syntonic comma. An octave consists of twelve of these generating intervals, a perfect fifth contains seven, and a major third contains four. The layout-mapping can be either $\mathbf{L} = 1$ or $\mathbf{L} = -1$. If the 12-TET generating interval is layout-mapped to +1 the notes go sequentially up in pitch from left to right, which corresponds to a linear keyboard as shown in Figure 4.

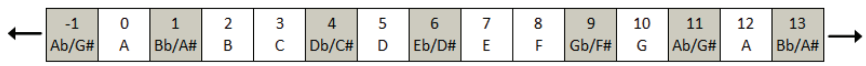

Figure 4: The one-dimensional keyboard layout $\mathbf{L} = 1$ in 12-TET.

The tuning range of rational invariance for this temperament is small. Increasing the size of the generator by just two cents increases the size of the octave by 24 cents. Though the piano is frequently tuned with stretched octaves, this stretching is typically less than about half a cent on the generating interval (about six cents per octave). The tuning range of ordinal invariance is unbounded since no amount of retuning changes the order of the notes. For this case, the ordinal range is not a useful measure.

Due to the isomorphic layout-mapping, this layout has the property that any rationally identifiable interval or chord (such as the major triad) is fingered the same for any position on the button-row. Starting on any note (for instance C), a major triad is played using the buttons four and seven steps to the right. The geometric shape of a major triad is, therefore, always (0)–(4)–(7) for all transpositions. Similarly a melody is fingered the same wherever it is located on the button row. Starting on any note (for instance C), the melody Re→Mi→Fa→Fi→So is played using a button, then a button two steps to the right, then one more step to the right, then one more step to the right, then one more step to the right. The geometric shape of this melody is, therefore, always (0)→(2)→(3)→(4)→(5) for all transpositions.

A different rank-one temperament, which also tempers out the syntonic comma (amongst others) is 19-TET, as shown in Figure 5. This temperament has a single generating interval α of $2^{1/19}$ (≈ 63.2

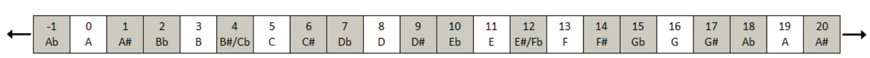

Figure 5: Note positions of 19-TET using $\mathbf{L} = 1$.

cents), and the octave consists of 19 of these generating intervals. The perfect fifth is 11 generating intervals wide, the major third is 6 generating intervals wide. Like 12-TET, it has a small range of rational tuning invariance and a similar transpositional invariance. However, the fingering of both the major triad (0)→(6)→(11) and the Re-Mi-Fa-Fi-So melody (0)→(3)→(5)→(6)→(8) are different from the 12-TET design in Figure 4. This shows concretely how the linear one-dimensional design fails to be tuning invariant.

Layout-mapping from a Rank-two Temperament to a Button-row

A layout-mapping is invertible by definition, so a layout-mapping from a rank-two regular temperament to a one-dimensional button-row must be nonlinear. As established above, a nonlinear mapping breaks transpositional invariance. A common example of a nonlinear layout-mapping is a piano-style layout, which takes a finite subset of the notes produced by a rank-two meantone, (such as quarter-comma) and layout-maps them in pitch order to a twelve note per octave keyboard, as illustrated in Figure 6. The physical/geometrical irregularity of the piano keyboard, interpreted as a row of black keys interspersed with a row of white keys, is collapsed here to a single row for the sake of simplicity.

←	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	→
	G#	A	Bb	B	C	C#	D	Eb	E	F	F#	G	G#	A	Bb	

Figure 6: Nonlinear piano-style layout.

Such layout-mappings (as applied to a standard, geometrically irregular keyboards) were used in the seventeenth century [Barbour, 1951], though sometimes A♭ was used in place of G♯. In quarter-comma (or any other meantone tuning requiring two generating intervals), major thirds (e.g. C–E) are tuned differently to diminished fourths (e.g. C♯–F). In the nonlinear layout of Figure 6, going up four steps sometimes produces a major third and sometimes produces a diminished fourth, thus breaking transpositional invariance. For example, starting from C and going up four steps to E produces a Just major third of size 386 cents, which is clearly identifiable as 5/4. On the other hand, starting at C♯ and going up four steps to F produces a diminished fourth with 427 cents. This is a wolf interval that is unlikely, in this context (where there exist more closely tuned major thirds), to be rationally identified as 5/4.

The only way to gain transpositional invariance for this layout is to tune the fifths to $2^{7/12}$, which is the only (octave-reduced) tuning that does not differentiate between major thirds and diminished fourths. This is consistent with the theorem because when the fifth is tuned to $2^{7/12}$ and the octave is tuned to 2, the two generating intervals are no longer multiplicatively independent. In this case, the rank of the temperament has collapsed to one and it is equivalent to 12-TET. A similar argument shows that for a button-row with N notes per octave, any tuning that produces an n-TET where $n = N$ will have transpositional invariance, but for this tuning only.

Layout-mapping from a Rank-two Temperament to a Button-field

One solution to the above problems is to use a two-dimensional button-field so that the layout-mapping from rank-two meantone to the button lattice is isomorphic. The simplicity afforded by the invariant fingering of an isomorphic layout is illustrated in the following examples using the Thummer's default Wicki layout.

To be concrete, Wicki is a layout-mapping to a hexagonal button-field, such that the temperament's two characteristic generating intervals α and β are layout-mapped to the generating vectors $\psi = (0, 2)'$ and $\omega = (1, 1)'$, i.e., $\mathbf{L} = \left(\begin{smallmatrix} 0 & 1 \\ 2 & 1 \end{smallmatrix}\right)$. This layout provides a good balance of octave-reduced intervals versus overall octave range for relatively small button-fields (such as the QWERTY computer keyboard) when using meantone tunings, and it also functions well over many non-meantone temperaments. Additionally, the Wicki layout allows a wide range of pitches to be played with a single hand, though it has lost the simple relationship between pitch height and location on the keyboard, as may be found, for instance, in the design of [Bosanquet, 1877]. The basic principles illustrated in the following examples may also be applied to other regular temperaments, MOS scales, and layouts.

Figure 7 shows the default positions of the pitches on the Thummer, and superimposed are the symbols of the QWERTY keyboard (in the middle four rows) to show how a standard computer keyboard can also serve as a Wicki layout. When the generating intervals α and β are an octave and a fifth, the standard octave-reduced diatonic intervals fall in the pattern shown in Figure 8.

Tuning Range of Ordinal Invariance

Assuming that the generating intervals are $\alpha = 2$ (an octave), and $\beta = F$ (an alterable tempered perfect fifth), the pitch order of buttons on the Wicki layout has seven different configurations as the tuning of F moves across the valid tuning range. Figure 9 shows the pitch order of the buttons, starting from

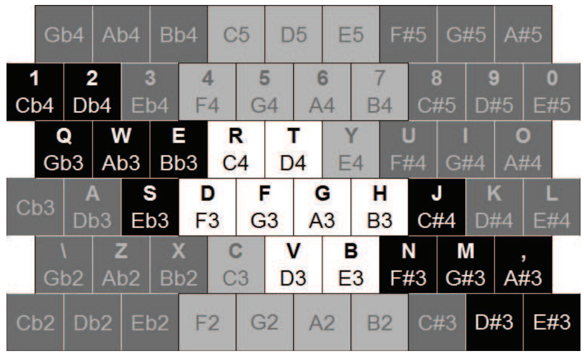

Figure 7: The position of notes and QWERTY using the Wicki layout.

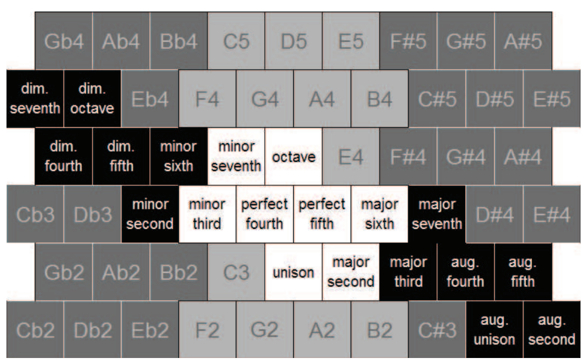

Figure 8: The position of the regular diatonic intervals using the Wicki layout; intervals are given with respect to an arbitrarily chosen root key labeled unison.

D3 (numbered 0) up to D4 (an octave above), as F traverses the $2^{4/7} < F < 2^{3/5}$ range. Despite these different configurations, every melodic interval that is "legal" to common practice has its order preserved (intervals such as G♯↔A♭, G♭↔E♯, or C♭↔B♭ typically have no melodic function in common practice music). For example, observe that although the pitch order changes as the tuning is raised from $2^{4/7}$ to $2^{3/5}$, the diatonic notes (colored white) remain in the same order, and so do the chromatic intervals. The shape of the chromatic melody Re→Mi→Fa→Fi→So is indicated by crosses, and the pitch order within this shape remains the same, only breaking down when $F \leq 2^{4/7}$ or $F \geq 2^{3/5}$.

Tuning Range of Rational Invariance

Figure 10 shows the recognizable harmonic consonances (according to common practice) that are signified over a range of values of the fifth F. This example is again centered on the note D3 (an arbitrary choice). A natural consequence of invariant fingering is that the steps of various n-TETs automatically line up to the correct button position. This is illustrated by Figure 11, which shows a selection of meantone n-TETs.

The (approximate) range of tunings maintaining invariant identification of harmonic and melodic intervals for the meantone temperament and the diatonic/chromatic scale are shown in Figure 2 (which is compiled from information in [Barbour, 1951], [Huygens-Fokker], [Lorentz, 2001], [Sethares, 2004], and [Wickipedia, 2006b]). A performer can maintain the same fingerings of chords and melodies throughout the relevant range of this tuning continuum. Note that Figure 2 indicates only the meantone temperament, but a given tuning can belong to more than one regular temperament. For example, 53-TET tuning falls within the valid tuning ranges of both the meantone and the schismatic temperament (that tempers the 5-limit generators of Equation (1) with $a = 11$, $b = -4$, and $c = -2$. Both approaches map all the primary consonances well within the range of rational (as well as ordinal) identification.

Dynamic (Re)tuning

Keyboard designs that are invariant in both transposition and tuning offer several performance possibilities to the computer-based musician and composer. With a single control parameter (perhaps triggered by a modulation wheel, data slider, thumb–operated joystick, or MIDI controller), it is possible to navigate through the various tunings of Figures 2 and 9–11.

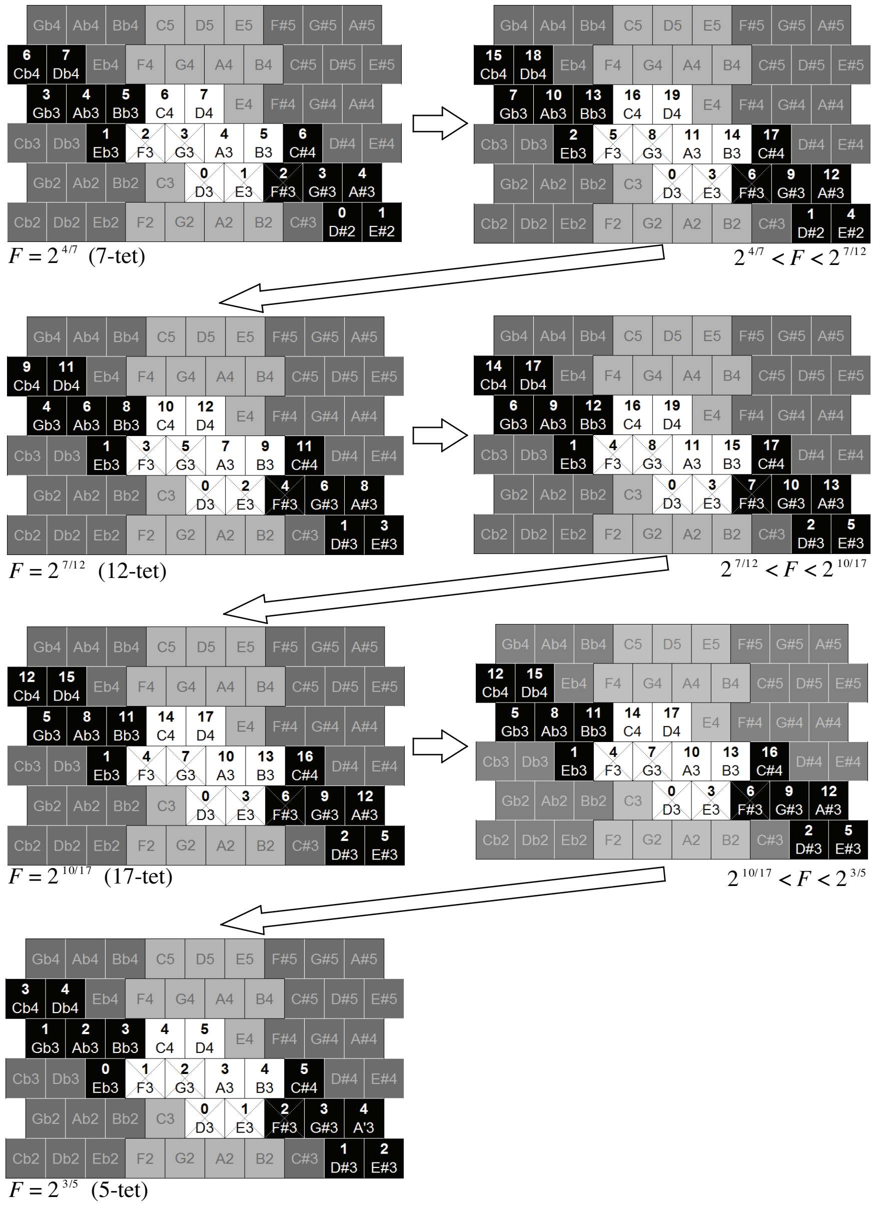

Figure 9: Keyboard layouts for selected tunings within the syntonic tuning continuum are shown over the full valid tuning range, which corresponds to the range of recognizable diatonic tunings.

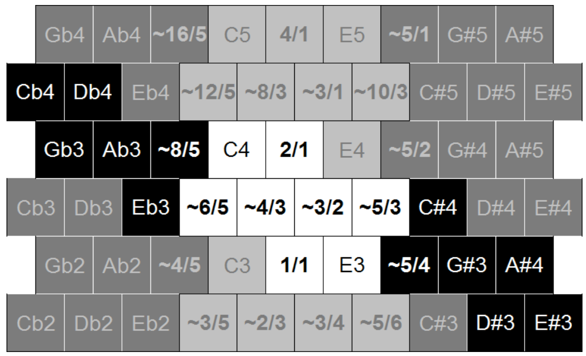

Figure 10: Common practice harmonic consonances for the meantone temperament.

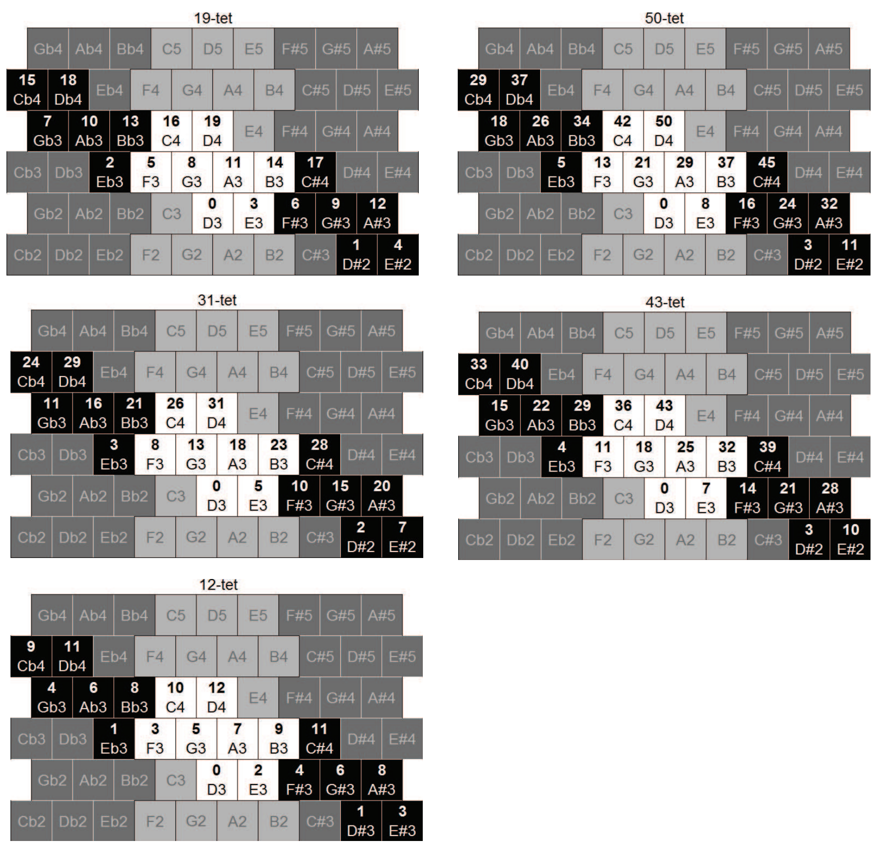

Figure 11: The step positions for a selection of meantone n-TETs.

For example, one might choose to perform a simple I–IV–V–I pattern such as shown in Figure 12. The three staves show the score as it would be written in standard 12-TET. The numerical annotations show the scale steps in each of three tunings (5-TET, 22-TET, and 7-TET) as the parameter β moves from $2^{3/5}$ to $2^{13/22}$ to $2^{4/7}$. As shown in Figure 9, the fingering of the chords remains the same throughout the continuum. Thus players of a generalized keyboard which is tuning invariant may be able to transfer competence in one tuning (such as 12-TET) to others; hard-won manual dexterity can be transfered directly to other tunings within the continuum.

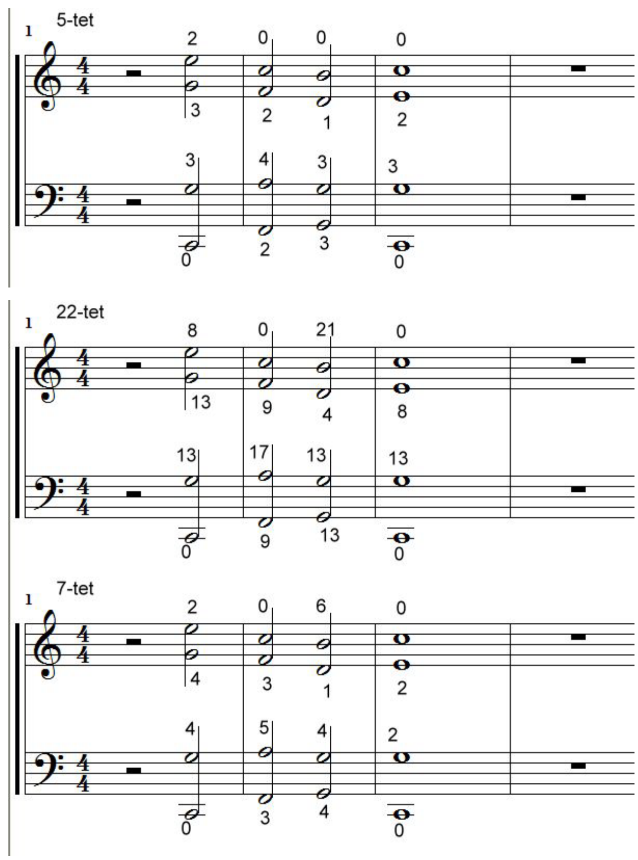

Figure 12: A simple chord pattern annotated with the scale-step numbers in 5-TET, 22-TET, and 7-TET. These (and all others from the tuning continuum in Figure 2) can be played with identical fingerings on a keyboard that is invariant in transposition and in tuning.

The tuning can also be changed dynamically during performance. By analogy with pitch bend, this might be called "tuning bend," where the exact tuning of each note in each interval is changed in response to the physical motion of the controller. For example, the performer might push up into a Pythagorean tuning to give melodies more expressive power, and then drop down to quarter-comma for more pleasing triads. This may give keyboard players greater flexibility in mimicking the kinds of expressive pitch deviations that string (and aerophone) players reveal when altering their intonation phrase by phrase [Sundberg, 1989]. Moreover, tuning bends are not limited to monophonic implementations. Rocking the controller back-and-forth describes a kind of (polyphonic) vibrato where each note follows its own trajectory through pitch space.

Conclusions

This paper has described the temperament-mapping from an underlying JI system to a lower rank regular tuning system and the layout-mapping that transforms the intervals of that temperament to a button lattice. One implication of the analysis is that a rank-two temperament requires a two-dimensional button field in order to be invariant in both transposition and tuning. The benefits of invariant fingerings as offered by isomorphic button-fields (such as the Thummer and the keyboard designs of Bosanquet, Fokker, and others [Keisler, 1988]) occur in both musical education and performance.

Despite its widespread success, the piano-style keyboard is incapable of achieving the advantages of an isomorphic keyboard, for two reasons. First, it is essentially one–dimensional and so can only have transpositional invariance for equal temperaments. Second, because it is irregularly shaped, even the potential transpositional invariance of 12-TET is geometrically distorted. This means that the same chord, or the same melody, is fingered differently and has a different shape depending on where it is played on the keyboard. This provides a barrier to learning, because every different key needs to be learned by rote, and the underlying regularities (rational and ordinal invariances) are hidden.

In contrast, isomorphic button-fields have same-shape chords and melodies at all locations on their playing surface (except, of course, at the edges where the regularities must break down). Furthermore, the structure of the button-field is a geometrical projection of 5-limit Just Intonation, which some theorists have viewed as the musical source from which the identification and meaning of common intervals is derived. Generalized isomorphic keyboards capable of tuning and transpositional invariance may thus help to remove some of the barriers to the exploration of new tuning systems.

Of course, there are also disadvantages to such general keyboards. Simple diatonic performance (such as occurs with the white keys of a piano in the key of C) becomes more complex. There is no longer a simple linear relationship between pitch height and location on the keyboard. Finally, the standard keyboard has the advantage of familiarity to a large installed user base.

Many of the individual tunings within the syntonic continuum have been used historically (such as the various meantones), have been used cross culturally (for example, 7-TET provides a close approximation to traditional Thai music [Morton, 1980] and 5-TET is close to Indonesian slendro [Surjodiningrat, et. al., 1993]), and are being explored with the help of electronic musical instruments. Generalized keyboards with tuning invariance allow easy navigation of the tuning continuum and easy performance throughout a variety of such tunings. For example, an appropriate tuning can be quickly chosen for the performance of baroque chamber music (e.g. quarter-comma), or medieval ars nova (Pythagorean), or modern atonal (12-TET). Finally, the possibilities of dynamically retuning throughout the continuum offers unique expressive potential.

References

[Barbour, 1951] Barbour, J. M., *Tuning and Temperament*, Da Capo Press.

[Blackwood, 1985] Blackwood, *The Structure of Recognizable Diatonic Tunings*, New Jersey, Princeton University Press.

[Bosanquet, 1877] Bosanquet, R. H. M., *Elementary Treatise on Musical Intervals and Temperament*, cited in Helmholtz (1877).

[Carey, 1989] Carey, N. and Clampitt, D., Aspects of Well-Formed Scales. *Music Theory Spectrum*, 11, 187-206.

[Carlos, 1987] Carlos, W., "Tuning: at the crossroads," *Computer Music Journal*, Spring, 29-43.

[Chandler 2001] Chandler, D., *Semiotics: The Basics*, Routledge.

[Clough, 1999] Clough, J., N. Engebretsen, and J. Kochavi, "Scales, sets, and interval cycles," *Music Theory Spectrum* 10: 19-42.

[Fokker, 1955] Fokker, A., Equal Temperament and the Thirty-one-keyed Organ. Available online at: www.xs4all.nl/∼huygensf/doc/fokkerorg.html (accessed 26 March 2007).

[Grady, 1999] Grady, K., "CS," http://launch.groups.yahoo.com/group/tuning/message/5244.

[Helmholtz, 1877] Helmholtz, H. L. F., *On the Sensations of Tones*. Trans. Ellis, A.J.. Reprint by Dover in 1954 (New York: Dover).

[Huygens-Fokker] Huygens-Fokker Foundation. n.d. "Huygens-Fokker Foundation," http://www.xs4all.nl/∼huygensf/.

[Keisler, 1988] Keislar, D., "History and Principles of Microtonal Keyboard Design," Report STAN-M-45, Stanford University.

[Lorentz, 2001] Lorentz, R., "19-TET in a Renaissance Chanson by Guillaume Costeley," Conference and Festival of Music in Alternate Tunings, Microfest 2001: Paper Session 1, April 6.

[McLaren, 1991] McLaren, B., "General methods for generating musical scales," Xenharmonikon 13.

[Meantone Citation] Citation for syntonic comma gives meantone temperaments

[Milne, 2006] Milne, A. J., http://www.tonalcentre.org/Deepertheory.html

[Morton, 1980] D. Morton, "The Music of Thailand," in E. May, ed., *Music of Many Cultures*, University of California Press, Inc., Berkeley, CA (1980)..

[Sethares, 2007] Sethares, W.A., Milne, A. and Plamondon, J., "Consistent fingerings for a continuum of syntonic commas," Presentation given at American Mathematical Society 2007, New Orleans, LA, 6 Jan.

[Sethares, 2004] Sethares, W.A., *Tuning, Timbre, Spectrum, Scale*. (Springer-Verlag. Second edition 2004).

[Smith, 2006] Smith, G.W., Xenharmony, Available online at: bahamas.eshockhost.com/∼xenharmo/ (accessed October, 2006).

[Sundberg, 1989] Johan Sundberg, Anders Friberg and Lars Frydén, "Rules for automated performance of ensemble music," *Contemporary Music Review,* 3:1, 89–109.

[Surjodiningrat, et. al., 1993] W. Surjodiningrat, P. J. Sudarjana, and A. Susanto, *Tone Measurements Of Outstanding Javanese Gamelan In Yogyakarta And Surakarta,* Gadjah Mada University Press (1993).

[Tenney, 1985] Tenney, J., *A History of 'Consonance' and 'Dissonance'*, New York: Excelsior Music Pub. Co.

[Thumtronics, 2007] Thumtronics Ltd., Thumtronics: The New shape of Music. Available online at: www.thummer.com (accessed 26 March 2007).

[Wicki, 1896] Wicki, K., "Tastatur für Musikinstrumente," Swiss patent Nr. 13329, in Neumatt (Münster, Luzern, Schweiz).

[Wilson, 2007] Wilson, E., "Bosanquet – A Bridge – A Doorway to Dialog," Available online at www.anaphoria.com/xen2.PDF (accessed 26 March 2007).

[Wickipedia, 2006a] Wikipedia, "Lattice (group)," http://en.wikipedia.org/wiki/Lattice_(group) (accessed 11 October, 2006).

[Wickipedia, 2006b] Wikipedia, "53 Equal Temperament," http://en.wikipedia.org/wiki/53_equal_temperament (accessed 11 October, 2006).

[Wilson, 1975] Wilson, E., Letter to Chalmers Pertaining to Moments Of Symmetry/Tanabe Cycle. Available online at: www.anaphoria.com/mos.PDF (accessed 26 March 2007).

A Proof of Theorem 1

Let $f : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$ be an invertible mapping from the 2-D generators of a rank 2 temperament to the 2-D button layout. Any note N in the temperament can be represented in terms of its generators α and β as $N = \alpha^n \beta^m$ which can be written as the vector $\begin{pmatrix} n \\ m \end{pmatrix}$ for $n, m \in \mathbb{Z}$. The note is located on the keyboard at $f(N) = f\begin{pmatrix} n \\ m \end{pmatrix}$. An interval I is a ratio of two notes

$$\frac{N_1}{N_2} = \frac{\alpha^{n_1}\beta^{m_1}}{\alpha^{n_2}\beta^{m_2}}$$

which can be written $\begin{pmatrix} j \\ k \end{pmatrix}$ where $j = n_1 - n_2$ and $k = m_1 - m_2$. A layout is *transpositionaly invariant* if every fixed interval I is fingered in the same manner, that is, if

$$f(N_1) - f(N_2) = f(N_3) - f(N_4) \tag{2}$$

whenever

$$\frac{N_1}{N_2} = \frac{N_3}{N_4} = I. \tag{3}$$

Thus transpositional invariance requires that the difference in locations between notes on the keyboard depends only on the interval (the j and k) and not on the particular notes (the n_i and m_i). Expanding (2) yields

$$f\begin{pmatrix} n_1 \\ m_1 \end{pmatrix} - f\begin{pmatrix} n_1 + j \\ m_1 + k \end{pmatrix} = f\begin{pmatrix} n_3 \\ m_3 \end{pmatrix} - f\begin{pmatrix} n_3 + j \\ m_3 + k \end{pmatrix}$$

which has used the equality of the intervals from (3), i.e., that $n_2 = n_1 + j$, $m_2 = m_1 + k$, $n_4 = n_3 + j$, and $m_4 = m_3 + k$. If the mapping f is linear, then this can be rewritten

$$f\begin{pmatrix} n_1 \\ m_1 \end{pmatrix} - f\begin{pmatrix} n_1 \\ m_1 \end{pmatrix} + f\begin{pmatrix} j \\ k \end{pmatrix} = f\begin{pmatrix} n_3 \\ m_3 \end{pmatrix} - f\begin{pmatrix} n_3 \\ m_3 \end{pmatrix} + f\begin{pmatrix} j \\ k \end{pmatrix}$$

which collapses to an identity for any n_i and m_i, demonstrating that linear layout mappings are transpositionaly invariant. An analogous argument works for any invertible rank r mapping $f : \mathbb{Z}^r \to \mathbb{Z}^r$.

B Proof of Theorem 2

The converse of theorem 1 is demonstrated by showing that transposition invariance implies linearity of f. This will first be shown in the scalar case, where the defining equation (2) for transposition invariance is that for all fixed intervals i,

$$f(x + i) - f(x) = f(y + i) - f(y) \tag{4}$$

for every $x, y \in \mathbb{Z}$. We also assume that $f(0) = 0$. Linearity of f is shown by demonstrating additivity and homogeneity.

Additivity: Since (4) holds for all x, y, it must hold for $x = j$ and $y = 0$. Substituting into (4) shows $f(j+i)-f(j) = f(0+i)-f(0)$. Since $f(0) = 0$, this can be rearranged to show $f(i+j) = f(i)+f(j)$.

Homogeneity: Since (4) holds for all x, y, it must hold for $x = i$ and $y = 0$. Substituting into (4) shows $f(i + i) - f(i) = f(0 + i) - f(0)$. Since $f(0) = 0$, this can be rearranged to show $f(2i) = 2f(i)$. Induction can be used to show the general case: suppose that $f((k-1)i) = (k-1)f(i)$. Let $x = (k-1)i$ and $y = 0$. Substituting into (4) shows $f((k - 1)i + i) - f((k - 1)i) = f(i)$ Using the inductive hypothesis, this can be rearranged to show $f(ki) = (k - 1)f(i) + f(i) = kf(i)$.

The generalization to two (or n) dimensions is straightforward.